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Recursively-Defined Combinatorial Functions: The Case of Binomial and Multinomial Coefficients and Probabilities

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Authors' contributions

This work was carried out in full collaboration between the two authors. Author AMAR envisioned, designed and structured the study, performed the recursive and iterative analysis, solved the numerical examples, managed the literature survey and wrote the preliminary manuscript. Author MAA contributed to the literature search, implemented the algorithms, drew the figures, and made computational comparisons. Both authors read and approved the final manuscript.

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Abstract

This paper studies a prominent class of recursively-defined combinatorial functions, namely, the binomial and multinomial coefficients and probabilities. The paper reviews the basic notions and mathematical definitions of these four functions. Subsequently, it characterizes each of these functions *via* a recursive relation that is valid over a certain two-dimensional or multi-dimensional region and is supplemented with certain boundary conditions. Visual interpretations of these characterizations are given in terms of regular acyclic signal flow graphs. The graph for the binomial coefficients resembles a Pascal Triangle, while that for trinomial or multinomial coefficients looks like a Pascal Pyramid, Tetrahedron, or Hyper-Pyramid. Each of the four functions is computed using both its conventional and recursive definitions. Moreover, the recursive structures of the binomial coefficient and the corresponding probability are utilized in an iterative scheme, which is substantially more efficient than the conventional or recursive evaluation. Analogous iterative evaluations of the multinomial coefficient and probability can be constructed similarly. Applications to the reliability evaluation for two-valued and multi-valued k-out-of-n systems are also pointed out.

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1 Introduction

Many combinatorial functions can be characterized by a general framework based on simple two-dimensional or multi-dimensional recursion [1-19]. Four prominent cases among these combinatorial functions are the functions of binomial coefficients [6,10,15,20-22] and binomial probabilities [1,6,10,23,24], as well as their extensions to multinomial coefficients [21,22,25-32] and multinomial probabilities [21,22,33,34].

This paper reviews the basic notions and concepts of the aforementioned four combinatorial functions. It points out that computing any of them (*via* conventional methods involving factorials) has exponential temporal complexity. The paper explores the recursive structure for each of these functions with an eye of constructing *more efficient iterative* computational methods, which could possibly be of a polynomial (sub-exponential) rather than exponential temporal complexity.

We follow the technique introduced by Rushdi [1,6,10] of characterizing a recursively-defined function *via* three entities, *viz.*, (a) a recursive relation, (b) a region of validity for the recurrence, and (c) boundary conditions. We provide pictorial insight into these entities through the use of (Mason) linear signal flow graphs (SFGs) [35-37], which turn out to be regular and acyclic or loopless. Linear signal flow graphs are very useful in representing and manipulating linear relations in a wide variety of scientific and engineering applications [6,10,37-42]. The SFG for the binomial coefficient is simply a Pascal Triangle (also known as al-Karkhi Triangle or al-Khayyam Triangle) [21,22,43-48], while the SFG for the multinomial coefficients is a Pascal Pyramid or a Pascal Hyper-pyramid [49-56].

The topics of binomial coefficients and binomial probabilities, and their associated topics of k-out-of–n system reliability and unreliability have been extensively studied before [1,6,10,15,20-24]. They are included herein to make the paper self-contained and to facilitate the extension to the topics of multinomial coefficients and multinomial probabilities, and subsequently to the associated topics of multi-valued k-out-of-n system reliability and unreliability.

The organization of the remainder of this paper is as follows. Section 2 introduces binomial coefficients and probabilities and shows that their recurrences are represented by signal flow graphs that resemble Pascal Triangles. Section 3 presents and exposes the concept and conventional computational methods for the multinomial coefficients and probabilities. Subsequently, Sections 4 and 5 introduce recurrences for multinomial coefficients and multinomial probabilities, respectively, and show how such recurrences are represented *via* Pascal Pyramids or Hyper-pyramids. Section 6 makes a quick comparison of iteration versus recursion, with a stress on the utility of both techniques in the computation of the four functions considered herein. Section 7 concludes the paper.

2 Binomial Coefficients and Binomial Probabilities

In a (conventional) Bernoulli trial, one of two distinct outcomes might result. These two outcomes are typically called success and failure, and their probabilities are assumed to be constant, and denoted by p and q, respectively. The number of k successes in n independent trials is called the binomial (combinatorial) coefficient, and is given by [21-24]

$$B(k,n) = \frac{n!}{k!\,(n-k)!} = \prod_{m=1}^{MIN(k,\ n-k)} \left(\frac{n-m+1}{m}\right). \tag{1}$$

The binomial coefficient $B(k,n)$ also expresses the number of ways of selecting k objects out of n objects, when repetition (replacement) is not allowed and order does not matter. In particular, $B(k,n)$ denotes the number of subsets with k elements among the subsets of a set on n elements. The binomial coefficient satisfies the recursive relation [6,15,21,22].

$$B(k,n) = B(k,n-1) + B(k-1,n-1), \;\; 0 < k < n, \tag{2}$$

subject to the boundary conditions

$$B(k,n) = 1 \qquad for\ (k = 0\ or\ k = n) \qquad and\ \ n \geq 0. \tag{3}$$

Fig. 1 is a Signal Flow Graph (SFG) representing the computation of $B(k,n)$ *via* (2) and (3). To facilitate comparison with the multinomial case, we use a Cartesian grid (k_1, k_2) where $(k_1 \geq\ 0,\ k_2 \geq\ 0)$ to represent $B(k,n)$ as $B(k_1,\ k_1 + k_2)$ where $k_1 = k$ and $k_2 = n - k$. Fig. 1 is traditionally referred to as Pascal Triangle (albeit it is also known by many other names such as al-Karkhi Triangle or al-Khayyam Triangle [57,58]). This triangle occupies an octant of the $k - n$ plane or the $k_1 - k_2$ plane. It extends without bound as n goes to infinity, but it is otherwise perfectly bounded by the two straight arrows given in (3), *viz*., $(k = 0, n \geq 0)$ and $(k = n,\ n \geq 0)$.

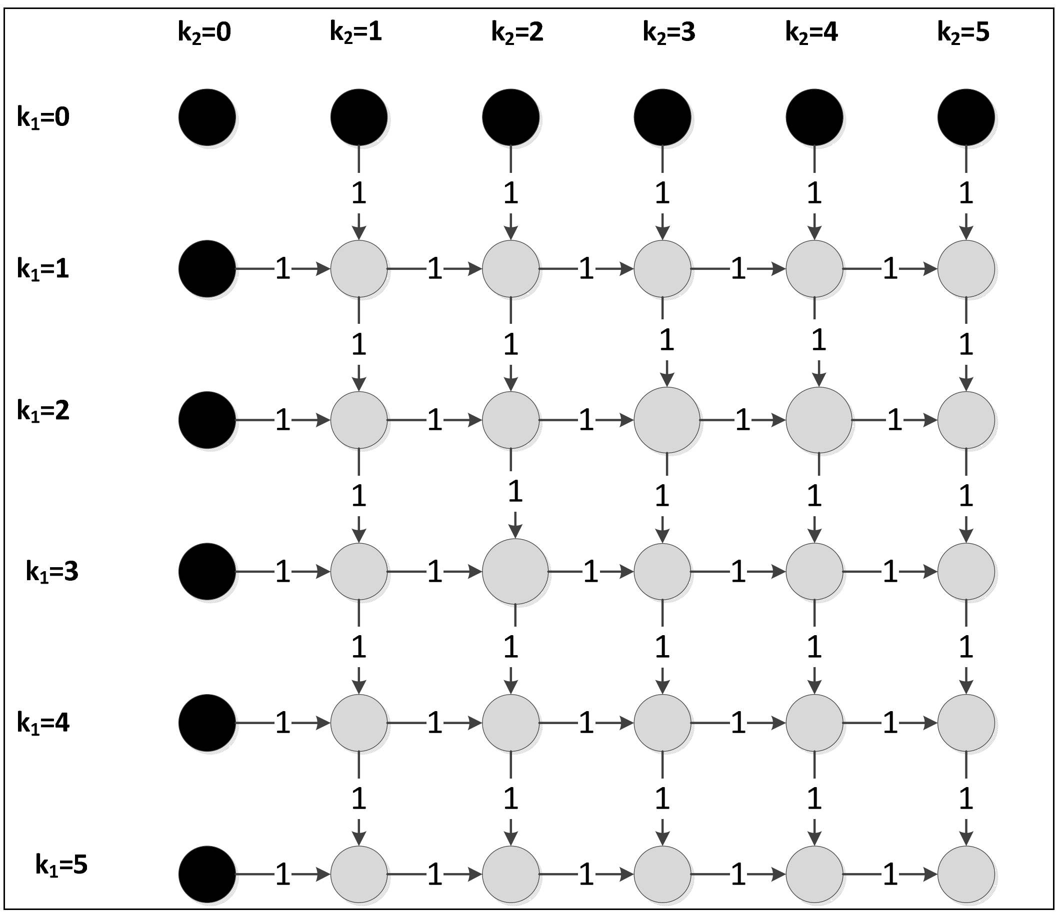

Fig. 1. Two Dimensional Signal Flow Graph for computing the binomial coefficients $B(k_1, k_1 + k_2)$. If only the top left triangular half of the shown square is retained, the figure becomes a Pascal Triangle (also called al-Karkhi Triangle or al-Khayyam Triangle). The value of a non-source node at $(k_1,\ k_2)$ is $B(k_1, k_1 + k_2)$ and equals the number of paths from source (black) nodes to this node. For example, the non-source nodes in the rightmost column shown are $B(1,6) = 6$, $B(2,7) = 21$, $B(3,8) = 56$, $B(4,9) = 126$, and $B(5,10) = 252$

For convenience, and to maintain symmetry, equations (1)-(3) are rewritten as:

$$B(k_1, k_1+k_2) = (k_1+k_2)!/\,(k_1!\,k_2!), \tag{1a}$$

$$\begin{aligned} B(k_1, k_1+k_2) &= B(k_1,\ k_1+k_2-1) + B\,(k_1-1, k_1+k_2-1), \\ &= B((k_1, k_1+k_2)|\,k_2 \leftarrow k_2-1) + B((k_1, k_1+k_2)|\,k_1 \leftarrow k_1-1), \\ &\qquad k_1 > 0, k_2 > 0, \end{aligned} \tag{2a}$$

$$B(k_1, k_1+k_2)=1\ for(\,k_1=0,\ k_2\ \geq 0)\ \ or\ \ (\,k_2=0, k_1\ \geq 0). \tag{2b}$$

Note that equations (2a) & (3a) exhibit symmetric behavior towards the two dimensions set by $k_1\ and\ k_2$. This symmetric behavior can be easily maintained when these two dimensions are replaced by r dimensions $(\,r > 2)$ in the study of multinomial coefficients. Note that the notation $(k_1, k_1+k_2)\ |\ k_i \leftarrow k_i - 1\ for\ i = 1, 2$ means that we substitute $k_i - 1$ for k_i in (k_1, k_1+k_2).

The probability of obtaining k successes in n trials is the probability mass function (pmf) of the binomial distribution [23,24], and is given by

$$E(k, n, p) = B(k, n)\, p^k\, (1-p)^{n-k}\,, \quad 0 \leq k \leq n, \tag{4}$$

while the corresponding Cumulative Distribution Function (CDF), and Complementary Cumulative Distribution Function (CCDF) are given by

$$U(k, n, p) = \textstyle\sum_{m=0}^{k-1}\ E\,(m. n, p). \tag{5}$$

$$R(k, n, p) = \textstyle\sum_{m=k}^{n}\ E\,(m. n, p). \tag{6}$$

The aforementioned $U(k, n, p)\ and\ R(k, n, p)$ represent, respectively, the unreliability and reliability of a (two-valued) k-out-of-n: G system, which is a coherent system that is said to be *"good"* if and only if *at least* k out of its n components are *"good"* [1,6,10].

Let us now generalize our notion of a Bernoulli trial by relaxing the condition that the probabilities $p\ and\ q$ of trial success and failure be constant (equivalently, assuming that component reliabilities are not necessarily equal). This generalization does not affect $B(k, n)$, but leads us to replace $E(k, n, p)$ by $E(k, n, \boldsymbol{p})$, where $\boldsymbol{p} = [p_1\ p_2\ p_3 \ldots\ p_n]^{T}$. Likewise, we replace $U(k, n, p)$ and $R(k, n, p)$ by $U(k, n, \boldsymbol{p})$ and $R(k, n, \boldsymbol{p})$. Note that equations (5) and (6) continue to hold with p replaced by $\boldsymbol{p}$, *i.e.,*

$$U(k, n, \boldsymbol{p}) = \textstyle\sum_{m=0}^{k-1}\ E\,(m. n, \boldsymbol{p}), \tag{5a}$$

$$R(k, n, \boldsymbol{p}) = \sum_{m=k}^{n}\ E\,(m. n, \boldsymbol{p}). \tag{6a}$$

The probability $E(k, n, \boldsymbol{p})$ satisfies the recursive relation [1-3, 6, 10, 13]

$$\begin{aligned} &E(k, n, \boldsymbol{p}) = q_n * E(k, n-1, \boldsymbol{p}/p_n) + p_n * E(k-1, n-1, \boldsymbol{p}/p_n)\ , \\ &0 \leq k \leq n, \quad \{k, n\} \neq \{0,0\}, \end{aligned} \tag{7}$$

where $\boldsymbol{p}/p_n = [\,p_1\, p_2\ p_3 \ldots\ p_{n-1}]^{T}$. Equations (7) are subject to the boundary conditions

$$E(k, n, \boldsymbol{p}) = 1.0\ \ for\ k = n = 0, \tag{8}$$

$$E(k, n, \boldsymbol{p}) = 0.0\ \ for\ (k = -1\ or\ k = n+1)\ \ and\ (n \geq 0). \tag{9}$$

Again, to facilitate extension to the multinomial case, we use the grid $(k_1,\ k_2)$ instead of the grid (k,n), where $k_1 = k\ and\ k_2 = n - k$, We also rewrite $p_n\ as\ p_{1n}$ and rewrite $q_n\ as\ p_{2n}$ where $p_{1n}\ and\ p_{2n}$ are the probabilities of outcome 1 and outcome 2 in the nth trial, respectively. With this change the vectors $\boldsymbol{p}$ and $\boldsymbol{p}/p_n$ are replaced by two column matrices of the form

$$\boldsymbol{p} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \dots\dots\dots & p_{1(n-1)} & p_{1n} \\ p_{21} & p_{22} & p_{23} & \dots\dots\dots & p_{2(n-1)} & p_{2n} \end{bmatrix}^T, \tag{10}$$

$$\boldsymbol{p}/\boldsymbol{p_n} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \dots\dots\dots & p_{1(n-1)} \\ p_{21} & p_{22} & p_{23} & \dots\dots\dots & p_{2(n-1)} \end{bmatrix}^T. \tag{10 a}$$

Equations (7) – (9) are now rewritten as

$$\begin{aligned} E(k_1, k_1 + k_2, \boldsymbol{p}) &= p_{2n} * E(k_1, k_1 + k_2 - 1, \boldsymbol{p}/\boldsymbol{p_n}) + p_{1n} * E(k_1 - 1, k_1 + k_2 - 1, \boldsymbol{p}/\boldsymbol{p_n}) \\ &= p_{2n} * E\ ((k_1, k_1 + k_2, \boldsymbol{p}/\boldsymbol{p_n}))|\ k_2 \leftarrow k_2 - 1)\ + \\ &\ p_{1n} * E\ \left((k_1, k_1 + k_2, \boldsymbol{p}/\boldsymbol{p_n})\right)|\ k_1 \leftarrow k_1 - 1), \end{aligned} \tag{7a}$$

$$E(k_1, k_1 + k_2, \boldsymbol{p}) = 1.0 \qquad for\ k_1 = k_2 = 0, \tag{8a}$$

$$E(k_1, k_1 + k_2, \boldsymbol{p}) = 0.0 \quad for\ (k_1 = -1, k_2 \geq 1)\ or\ (\ k_1 \geq 1, k_2 = -1). \tag{9a}$$

Again, equations (7a)–(9a) exhibit symmetric behaviour towards the two dimensions of $k_1\ and\ k_2$, and can keep such a behavior when a vectorial $\boldsymbol{k}$ of r dimensions $(r > 2)$ is involved.

Fig. 2 is a Signal Flow Graph (SFG) representing the computation of the binomial probability $E(k, n, \boldsymbol{p})$. This graph appeared earlier in a (k, n) grid in [1-3,6,10,13,16] and in a (k_1, k_2) grid in [19]. The SFG in Fig. 2 has a striking similarity to the one in Fig. 1. Both SFGs are acyclic (have no loops), and hence are analyzed through path enumeration and multiplication of transmittances along paths. For both graphs, the Mason gain formula [6,10,35-38] can be easily applied with all deltas set to 1.

The SFG in Fig. 1 has constant transmittances of value 1 each, while the one in Fig. 2 has variable transmittances representing outcome probabilities in a certain trial. Both graphs have an infinite number of source nodes along the two Cartesian axes, which serve as a boundary for the region in which the pertinent recursion is valid. However, Fig. 2 has only a single real source of value 1.0 (highlighted in black). It has a doubly infinite number of fictitious sources (each of value 0.0 coloured in white). These sources can be omitted, but they are retained herein to maintain the symmetry of the recursion and to appropriately depict the limit for its region of validity.

In passing, we note that the binomial recursion introduced in this section is just one kind of binomial recursion, which might be conveniently called Pascal's recursion, This recursion has the distinct advantages of being extendible to the multinomial case and of its use of addition solely (rather than multiplication and division). Other types of binomial recursion (that possess certain merits of their own) also exist [20, 59, 60], but will not be pursued further herein.

3 Multinomial Coefficients and Multinomial Probabilities

Let us generalize the original Bernoulli trials of Section 2 in another direction by allowing the number of outcomes in a single trial to be r > 2. We use the vector

$$\mathbf{k} = \ [k_1\ k_2\ \dots\ k_r\ ]^{\mathrm{T}},$$

to denote the numbers of occurrences of the various r outcomes in n independent trials, *i.e.*, k_i is the number of times outcome i (1 ≤ i ≤ r) occurs, so that

$$k_i \geq 0 \ , \quad 1 \leq i \leq \text{r}, \tag{11a}$$

$$\sum_{i=1}^{r} k_i = n. \tag{11b}$$

The multinomial coefficient M(**k**, n, r) is the number of ways in which outcome i $(1 \leq i \leq \text{r})$ occurs k_i times in n independent trials. It is given in terms of the factorial of n divided by the product of factorials of the k_i's, namely [21,22]

$$\text{M}(\mathbf{k}, \text{n}, \text{r}) = n!/(k_1!\ k_2! \ldots\ k_r!) = (k_1 + k_2 + \cdots + k_r)!\ /\ (k_1!\ k_2!\ \ldots\ k_r!). \tag{12}$$

Note that since B(k, n) = $\text{M}(\mathbf{k}, \text{n}, 2)$, equation (12) reduces to (1a) when $r = 2$.

Let us further assume that the probability of occurrence of outcome i throughout the n independent trials is a constant p_i such that

$$p_i \geq 0, 1 \leq i \leq \text{r}, \tag{13a}$$

$$\sum_{i=1}^{r} p_i = 1.0. \tag{13b}$$

The multinomial probability (the probability that occurrences of the r outcomes in n independent trials follow the vector $\boldsymbol{k}$) is denoted by $E(\boldsymbol{k}, n, r, \boldsymbol{p})$ and given by

$$E(\boldsymbol{k}, n, r, \boldsymbol{p}) = M(\boldsymbol{k}, n, r) \quad \textstyle\prod_{i=1}^{r} \ {p_i}^{k_i}, \tag{14}$$

We now further combine the two generalizations by allowing r > 2 outcomes and allowing variable probabilities for various outcomes in different trials. Let p_{ij} be the probability of occurrence of outcome i in trial j, so that

$$p_{ij} \geq 0\ ,\ 1 \leq i \leq \text{r}\ ,\ 1 \leq j \leq \text{n}, \tag{15a}$$

$$\sum_{i=1}^{r} p_{ij} = 1.0\ , \qquad 1 \leq j \leq \text{n}. \tag{15b}$$

This change does not affect $M(\boldsymbol{k}, n, r)$, but it affects $E(\boldsymbol{k}, n, r, \boldsymbol{p})$ since $\boldsymbol{p}$ is no longer a column vector of length r, but is instead a matrix of r rows and n columns. Equation (14) is no longer valid, and is to be replaced by a recursive relation to be presented in Section 5.

4 Recurrence for Multinomial Coefficients

To obtain a recurrence for the multinomial coefficients, we note that the r events

V_i = {outcome i occurs in the *nth* trial}, $1 \leq i \leq \text{r}$

are mutually exclusive and exhaustive. Hence, we can write (when $k_i > 0 \ \ for \ 1 \leq i \leq \text{r}$)

$$\text{M}(\mathbf{k}, \text{n}, \text{r}) = \textstyle\sum_{i=1}^{r} \quad (\text{M}(\mathbf{k}, \text{n}, \text{r})|\ \text{Vi}) = \quad \sum_{i=1}^{r} M(\boldsymbol{k}|k_i \leftarrow k_i - 1, n-1, r), \tag{16}$$

Note that the r-dimensional vector

$$\mathbf{k}=[k_1\ k_2\ \ldots.k_{i-1}\quad k_i\quad k_{i+1}\ \ldots.k_r]^{\mathrm{T}}\ . \tag{17}$$

is replaced in the right-hand side RHS of (16) by another vector of the same size given by

$$(\boldsymbol{k}|k_i \leftarrow k_i - 1) = [k_1\ k_2\ \ldots.k_{i-1}\quad (k_i - 1)\quad k_{i+1}\ \ldots.k_r]^{\mathrm{T}}. \tag{18}$$

The above recurrence occurs subject to the following boundary conditions, which are valid for n > 0

$$M(\boldsymbol{k}, n, r) = 1 \ \text{when}\ \ r = 1, \tag{19a}$$

$$M(\boldsymbol{k}, n, r) = M(\boldsymbol{k}/k_i, n\ , r - 1)\ \text{when}\ k_i = 0,\ 1 \le i \le \mathrm{r}. \tag{19b}$$

The condition in (19a) occurs when one particular $k_i = n$ so that all other $k_i's$ are 0's. The condition (19b) means that if the number of occurrences of a particular outcome becomes zero, this outcome is expelled from the set of possible outcomes, so that their number reduces from r to $(r - 1)$ and the r-dimensional vector of occurrences **k** in (17) is replaced by the $(r - 1)$-dimensional vector

$$\mathbf{k}/k_i=[k_1\ k_2\ \ldots.k_{i-1}\ \ k_{i+1}\ \ldots.k_r]^{\mathrm{T}}. \tag{20}$$

The recurrence in (16) and its boundary condition in (19) reduce for $r = 2$ to the recurrence in (2a) and its boundary conditions in (2b). Fig. 3 generalizes Fig. 1 for the trinomial case $(r = 3)$, and displays a SFG representing (16) and (19), in which the evaluation of M([1 1 1]$^{\mathrm{T}}$, 3, 3) is obtained recursively as the sum of three equal entities, namely

$$\mathrm{M}([1\ 1\ 1]^{\mathrm{T}}, 3, 3) = \mathrm{M}([0\ 1\ 1]^{\mathrm{T}}, 2, 3) + \mathrm{M}([1\ 0\ 1]^{\mathrm{T}}, 2, 3) + \mathrm{M}([1\ 1\ 0]^{\mathrm{T}}, 2, 3). \tag{21}$$

The first of the coefficients in the RHS of (21) is given as

$$\mathrm{M}([0\ 1\ 1]^{\mathrm{T}}, 2, 3) = \mathrm{M}([1\ 1]^{\mathrm{T}}, 2, 2) = \mathrm{M}([0\ 1]^{\mathrm{T}}, 1, 2) + \mathrm{M}([1\ 0]^{\mathrm{T}}, 1, 2). \tag{22}$$

Now, the two coefficients in the right-hand side RHS of (22) are given by

$$\mathrm{M}([0\ 1]^{\mathrm{T}}, 1, 2) = \mathrm{M}([1\ 0]^{\mathrm{T}}, 1, 2) = \mathrm{M}([1]^{\mathrm{T}}, 1, 1) = 1. \tag{23}$$

The above results mean that

$$\mathrm{M}([0\ 1\ 1]^{\mathrm{T}}, 2, 3) = 1+1= 2. \tag{24}$$

$$\mathrm{M}([0\ 1\ 1]^{\mathrm{T}}, 2, 3) = \mathrm{M}([1\ 0\ 1]^{\mathrm{T}}, 2, 3) =\mathrm{M}([1\ 1\ 0]^{\mathrm{T}}, 2, 3) = 2. \tag{25}$$

$$\mathrm{M}([1\ 1\ 1]^{\mathrm{T}}, 3, 3) =2 + 2 + 2=6. \tag{26}$$

Each of the values in (23)-(26) can be checked *via* the basic definition (12).

5 Recurrence for Multinomial Probabilities

The recursive relation for multinomial probability can be obtained *via* a straightforward extension of (7) or (7a) (for $k_i \ge 0$, and $\boldsymbol{k} \neq \boldsymbol{0}$)

$$E(\boldsymbol{k}, n, \boldsymbol{p}) = \sum_{i=1}^{r} \left(P_{in} * E(\boldsymbol{k}|k_i \leftarrow k_i - 1, n-1, \boldsymbol{p}/p_i) \right). \quad (27)$$

The above recurrence is subject to the following boundary conditions:

$$E(\boldsymbol{k}, n, \boldsymbol{p}) = 1.0 \;\; for \;\; \boldsymbol{k} = \boldsymbol{0} \;\; (n = 0 \; or \; \boldsymbol{p} = \emptyset = the \; empty \; matrix), \quad (28)$$

$$E(\boldsymbol{k}, n, \boldsymbol{p}) = 0.0 \;\; when \; one \; particular \;\; k_i = -1, \;\; n \geq 0. \quad (29)$$

Fig. 4 is a Signal Flow Graph (SFG) representing the computation of the trinomial probability $E(\boldsymbol{k}, n, \boldsymbol{p})$, where $\boldsymbol{p}$ is a 3-by-n matrix, whose columns (probabilities of outcomes in various trials) are distinct. Again, this SFG is acyclic (loopless). It has a striking similarity to the Pascal Pyramid in Fig. 3, and it is a 3-dimensional extension of Fig. 2. The SFG in Fig. 4 has a single real source of value 1.0 (highlighted as black) while the SFG in Fig.3 has a triply-infinite similar black nodes of value 1.0 each. In Fig. 4, a doubly-infinite number of fictitious zero nodes are assumed to exist at each of the three planes $k_1 = -1$, $k_2 = -1$ and $k_3 = -1$, However, these zero nodes are conveniently omitted in Fig. 4.

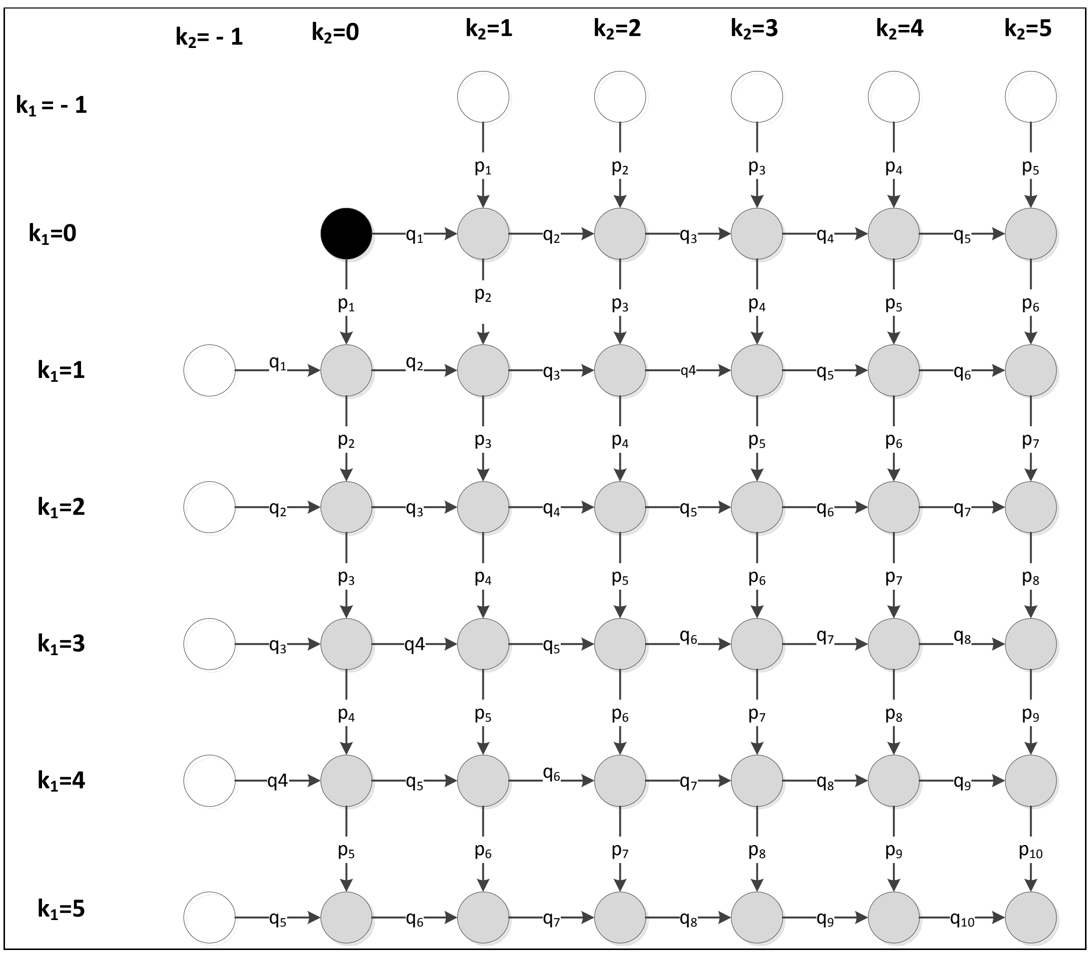

Fig. 2. Two-dimensional signal flow graph for the Probability Mass Function (PMF) of the binomial distribution

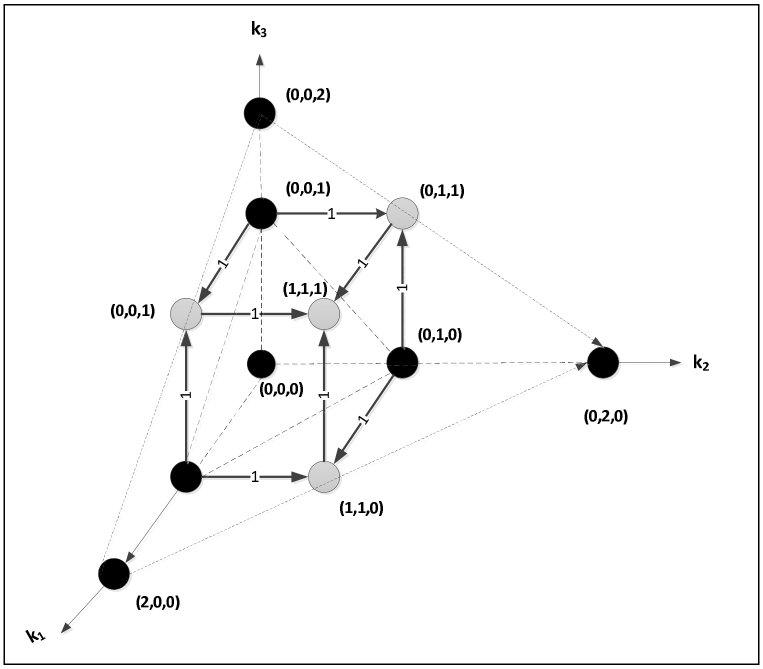

Fig. 3. Multi-dimensional signal flow graph for computing the trinomial coefficients. This is Pascal pyramid comprising Pascal triangles in various planes

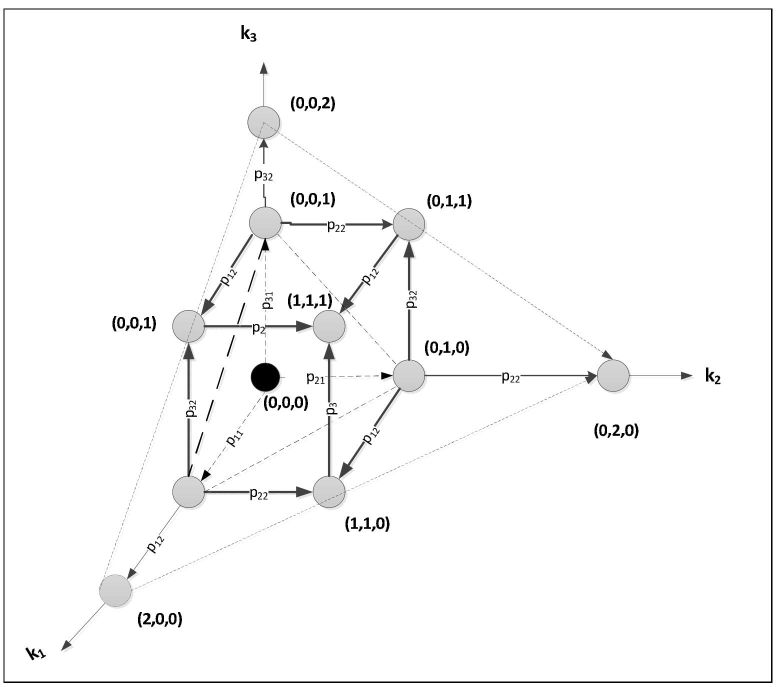

Fig. 4. Multi-dimensional signal flow graph for the probability mass function of the multinomial distribution up to two generalized Bernoulli trials. A doubly-infinite number of fictitious zero nodes exist at each of the planes $k_1 = -1$, $k_2 = -1$ and $k_3 = -1$ but are not shown

6 Iteration versus Recursion

Direct use of recursion constitutes a "lazy" way (from programming point of view) to compute any of the four quantities discussed herein. A recursive program might be written as follows (based on the assumption that applying the recurrence at a point in its region of validity does not continue indefinitely, but ultimately comes to an end at the boundary in a finite number of steps)

If (Conditions of the boundary are satisfied)
Then Use Boundary conditions
Else Utilize validity of recursion by implementing it and reducing the problem to two sub-problems for which the boundary is to be reached in fewer steps than those needed for the parent problem;

For example, the multinomial coefficient might be computed *via* the Matlab code

```
function S = multm(K)
    S = 0;
    sort(K);
    K(K==0) = [];
    n = sum(K);
    r = length(K);
    if r > 1
      for i=1:r
        X = K(i)-1;
        Y = K;
        Y(i) = X;
        Y(Y==0) = [];
        S = S+multm(Y);
      end
    else
      S = 1;
    end
  end


  k = input('Enter the elements of k: ');
  tic
  R = multm(k);
  toc
  disp('The multinomial is')
  disp(R)
```

Despite the elegant simplicity of the code above, it is a seriously inefficient algorithm for large values of n. This brings us to the heart of the problem of recursion versus iteration [1-3,5-7,10,20,23,24,61-72]. Fig. 5 pinpoints the main problem with recursion, with its unwarranted aggravation of both temporal and spatial complexities. In fact, use of recursion amounts to a need for constructing a recursion stack that needs a lot of time and memory as it continues to grow till each time the boundary is hit (at the leaves of the recursion tree [6]), upon which the stack growth is temporarily reversed, so that ultimately the stack diminishes with the required answer obtained. Fig. 5 demonstrates the recursion stack needed for the computation of $M([1\ 3\ 2]^T, 6, 3)$ with arrows denoting its growth and diminishing directions.

The Signal Flow Graphs represented herein are truly insightful for recursion elimination or removal [6,10, 73-80], *i.e.*, for replacing a recursive algorithm by a non-recursive or iterative one. Non-recursive algorithms already exist for the binomial coefficients [6,20] and for the binomial probability [1-3,6,10,23,24]. In these iterative algorithms, one can minimize computer storage by using a strategy of the row, column, or diagonal sweep in which the contents of a predecessor entity (row or column or diagonal) are safely overwritten by those of a corresponding successor entity [1,6,10]. One can also avoid or significantly reduce the danger of overflow encountered when computing binomial coefficients *via* factorials [24].

M ([1 3 2]^T,6,3) = M([0 3 2]^T,5,3) + M([1 2 2]^T,5,3) + M([1 3 1]^T,5,3)	=10 + 30 + 20=60
M([0 3 2]^T,5,3) = M([3 2]^T,5,2)	=10
M([1 2 2]^T,5,3) = M([0 2 2]^T,4,3) + M([1 1 2]^T,4,3) + M([1 2 1]^T,4,3)	=6+ 12 +12=30
M([0 2 2]^T,4,3) =M([2 2], T 4,2) = M([1 2] T,3,2) + M([2 1] T,3,2)	=3 + 3 =6
M([2 2] ^{T},4,2) = M([1 2] ^{T},3,2) + M([2 1], ^{T}3,2)	=3 + 3 =6
M([1 2] ^{T},3,2)	=3
M([2 1] ^{T},3,2) = M([1 1] ^{T},2,2) + M([2 0] ^{T},2,2)	=2+ 1 =3
M([1 1] ^{T},2,2) = M([0 1] ^{T},1,2) + M([1 0] ^{T},1,2)	=1 +1 =2
M([0 1] ^{T},1,2) = M([1] ^{T},1,1)	=1
M([1 0] ^{T},1,2) = M([1] ^{T},1,1)	=1
M([0 1 2]^T,3,3) = M([1 0 2]^T,3,3)	=3
M([1 1 1]^T,3,3) = M([0 1 1]^T,2,3) + M([1 0 1]^T,2,3) + M([1 1 0]^T,2,3)	=2+2+2=6
M([1 1 2]^T,4,3) = M([0 1 2]^T,3,3) + M([1 0 2]^T,3,3) + M([1 1 1]^T,3,3)	=3 +3+6=12
M([1 3 1]^T,5,3) = M([0 3 1]^T,4,3) + M([1 2 1]^T,4,3) + M([1 3 0]^T,4,3)	=4 +12+4=20

Fig. 5. Recursion Stack used in the computation of the multinomial coefficient M([1 3 2]^T, 6, 3). The downward arrow shows the direction of the initial growth of the stack, while the upward arrow indicates its subsequent decrease in size till ultimately it vanishes

In passing, we note that while the non-recursive algorithm for the binomial coefficient seems to have been known for millennia (definitely, for several centuries before Pascal), the analogous non-recursive algorithm for the (generalized) binomial probability seems to have appeared only recently. This latter algorithm is called the BH1 algorithm [1,6,10], and is deduced *via* generating functions by Barlow and Heidtmann [81]. It was later re-obtained *via* recursion removal by Rushdi [1,6,10] through an insightful development, in which SFGs played a pivotal role. The BH1 algorithm combined simplicity and efficiency (it has quadratic temporal complexity, and was believed for a while to be optimal). Later, Belfore [82] presented a more elaborate algorithm, which is currently believed to be the optimal non-recursive algorithm for the (generalized) binomial probability.

7 Conclusions

This paper offers a tutorial exposition of four recursively-defined combinatorial functions, *viz*., the binomial and multinomial coefficients and probabilities. Each of these functions is characterized by a certain recurrence over a specific region of validity together with given boundary conditions. These boundary conditions are along straight lines in a two-dimensional space for the binomial functions. Analogously, the boundary conditions extend along hyper-planes in the n-dimensional space for the multinomial functions. The region of validity is infinite as it extends without bounds when n tends to infinity. Otherwise, this region is bounded by the hyper-planes at which the boundary conditions hold.

A notable contribution of the paper is the visualization of the aforementioned recurrences and boundary conditions *via* acyclic Signal Flow Graphs (SFGs) in the two-dimensional or n-dimensional space. Each of the four functions is amenable to evaluation *via* recursive algorithms, which are elegant (albeit temporally and spatially inefficient). The SFG interpretation proved really handy for the construction of efficient iterative solution for the two binomial functions. Similar efficient iterative algorithms for computing the two multinomial functions are warranted and will be possibly constructed with the aid of the pertinent SFGs.

Competing Interests

Authors have declared that no competing interests exist.

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