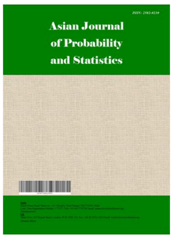

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Some Properties Via $\mathcal{G}_{\mathcal{N}}-$Preopen Sets in Grill Topological Spaces

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This work was carried out in collaboration between both authors. Author MAAM designed the study, performed the statistical analysis, and wrote the Article. Author AS managed the analysis of the study. The literature searches. Both authors read and approved the final manuscript.

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Abstract

This paper introduces and investigates the notions of $\mathcal{G}_{\mathcal{N}}-$disconnected sets, $\mathcal{G}_{\mathcal{N}}-$connected, $\mathcal{G}_{\mathcal{N}}-$precompact sets, and separation axioms via $\mathcal{G}_{\mathcal{N}}-$preopen sets. They will be introduced in grill topological spaces by using the class of $\mathcal{G}_{\mathcal{N}}-$preopen sets.

Keywords: Preopen sets; $\mathcal{G}_{\mathcal{N}}-$precompact; grill topological spaces.

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1 Introduction

The idea of grill on a topological space, given by Choquet [1], goes as follows: A non-null collection $\mathcal{G}$ of subsets of a topological spaces X is said to be a *grill* on X if

(i) $A \in \mathcal{G}$ and $A \subseteq B \Longrightarrow B \in \mathcal{G}$

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(ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Longrightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For a topological space X, the operator $\Phi : P(X) \rightarrow P(X)$ from the power set $P(X)$ of X to $P(X)$ was first defined in [2] in terms of grill; the latter concept being defined by Choquet [1] several decades back. Interestingly, it is found from subsequent investigations that the notion of grills as an appliance like nets and filters, turns out to be extremely useful towards the study of certain specific topological problems (see for instance [3], [4] and [5]). For a grill $\mathcal{G}$ on a topological space X, an operator from the power set $P(X)$ of X to $P(X)$ was defined in [6] in the following manner : For any $A \in P(X)$,

$$\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}, \text{ for each open neighborhood } U \text{ of } x\}.$$

Then the operator $\Psi : P(X) \rightarrow P(X)$, given by $\Psi(A) = A \cup \Phi(A)$, for $A \in P(X)$, was also shown in [6] to be a Kuratowski closure operator, defining a unique topology $\tau_{\mathcal{G}}$ on X such that $\tau \subseteq \tau_{\mathcal{G}}$. If (X, τ) is a topological space and $\mathcal{G}$ is a grill on X then the triple $(X, \tau, \mathcal{G})$ will be called a *grill topological space*.

In 1982 Mashhour [7], introduced the notion of a precontinuous function. In 2009 Al-Omari and Noiri [8], introduced the notions of $\mathcal{N}-$precontinuous function.

Under the notion of grill topological space and its operators above, several authors defined and investigated the notions in this part. In 2010 Hatir and Jafari [9], introduced the notions of G-precontinuous function.

This paper is organized as follows. In Section 2 we introduce the concept of $\mathcal{G}_{\mathcal{N}}-$disconnected sets and $\mathcal{G}_{\mathcal{N}}-$connected. Furthermore, the relationship with the other known sets will be studied. In Section 3 we introduce the concept of $\mathcal{G}_{\mathcal{N}}-$precompact sets. In Section 4 we provide new class of usual separation axioms, by using $\mathcal{G}_{\mathcal{N}}-$preopen sets, called ${}_{\mathcal{G}_{\mathcal{N}}} - T_2$ space, $\mathcal{G}_{\mathcal{N}}-$regular space (${}_{\mathcal{G}_{\mathcal{N}}} - T_3$ space) and $\mathcal{G}_{\mathcal{N}}-$normal (${}_{\mathcal{G}_{\mathcal{N}}} - T_4$ space).

For a topological space (X, τ) and $A \subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively. A subset A of a topological space X is called a preopen set, [7] if $A \subseteq Int(Cl(A))$. The complement of preopen set is called preclosed set. A subset A of topological space (X, τ) is called a $\mathcal{N}-$preopen set, [8] if for each $x \in A$, there exists a preopen set U_x containing x such that $U_x - A$ is a finite set. The complement of $\mathcal{N}-$preopen set is called $\mathcal{N}-$preclosed set. A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is called a $\mathcal{G}-$preopen set, [9] if $A \subseteq Int(\Psi(A))$. The complement of $\mathcal{G}-$preopen set is called $\mathcal{G}-$preclosed set.

Definition 1.1. [10] A topological space (X, τ) is called:
1. T_0 space if for two points $x \neq y \in X$, there is open set G in X such that $x \in G$ and $y \notin G$.
2. T_1 space if for two points $x \neq y \in X$, there are two open sets G and U in X such that $x \in G$, $y \notin G$, $y \in U$ and $x \notin U$.
3. T_2 space or Hausdorff space if for two points $x \neq y \in X$, there are two open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$.
4. Regular space if for each closed set F in X and each $x \notin F$, there are two open sets G and U in X such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called T_3 space if it is regular space and T_1 space.

Theorem 1.2. [10] A topological space (X, τ) is regular space if and only if for each $x \in X$ and for each open set N in X containing x, there is an open set M in X containing x such that $Cl(M) \subseteq N$.

Theorem 1.3. [10] A topological space (X, τ) is T_1 space if and only if $\{x\}$ is a closed set in X for all $x \in X$.

Definition 1.4. [9] A function $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ of a grill topological space $(X, \tau, \mathcal{G})$ into a space (Y, ρ) is called $\mathcal{G}-$precontinuous function if $f^{-1}(U)$ is $\mathcal{G}-$preopen set in $(X, \tau, \mathcal{G})$ for every open set U in Y.

Definition 1.5. [10] A topological space (X, τ) is called Urysohn space if for two points $x \neq y \in X$, there are two open sets G and U in X such that $x \in G$, $y \in U$ and $Cl(U) \cap Cl(G) = \emptyset$.

Definition 1.6. [8] A function $f : (X, \tau) \to (Y, \rho)$ is called $\mathcal{N}-$precontinuous function if $f^{-1}(U)$ is a $\mathcal{N}-$preopen set in X for every open set U in Y.

Definition 1.7. [7] A function $f : (X, \tau) \to (Y, \rho)$ is called *precontinuous function* if $f^{-1}(U)$ is a preopen set in X for every open set U in Y.

Theorem 1.8. [10] Let A and B be two subsets of a topological space (X, τ). If B is an open set in X then $Cl(A \cap B) = Cl(Cl(A) \cap B)$.

Definition 1.9. [10] A topological space (X, τ) is called:
1. An extremally disconnected space if the closure of every open subset of X is an open set in X.
2. Locally indiscrete space if every open set in X is a closed set.
3. $0-$dimensional space if it has a base consisting clopen sets.

Definition 1.10. [10] A topological space (X, τ) is called a disconnected space if it is the union of two nonempty subsets A and B such that $Cl(A) \cap B = \emptyset$ and $A \cap Cl(B) = \emptyset$.

Theorem 1.11. [10] A topological space (X, τ) is a disconnected space if and only if it is the union of two disjoint nonempty open subsets.

An open cover of a subset A of a topological space (X, τ) is a collection $C = \{G_\lambda : \lambda \in I\}$ of open subsets of X such that $A \subseteq \cup_{\lambda \in I} G_\lambda$, where I is an index set.

Definition 1.12. [10] A topological space (X, τ) is called a compact space if every open cover of X has finite subcover.

Definition 1.13. [8] A subset A of topological space (X, τ) is called strongly compact set in X if every preopen cover of A has a finite subcover.

Theorem 1.14. [9] Every $\mathcal{G}-$preopen set in a grill topological space $(X, \tau, \mathcal{G})$ is preopen set.

Theorem 1.15. [8] A subset A of topological space (X, τ) is strongly compact set in X if and only if every $\mathcal{N}-$preopen cover of A has a finite subcover.

2 $\mathcal{G}_\mathcal{N}-$Connected

Definition 2.1. Let $(X, \tau, \mathcal{G})$ be a grill topological space and A, B be two nonempty subsets of X. The sets A and B are called *a $\mathcal{G}_\mathcal{N}-$separated sets* if ${}_{\mathcal{G}_\mathcal{N}}Cl(A) \cap B = \emptyset$ and $A \cap {}_{\mathcal{G}_\mathcal{N}}Cl(B) = \emptyset$.

Remark 2.2. Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then
1. Any $\mathcal{G}_\mathcal{N}-$separated sets are disjoint sets, since $A \cap B \subseteq A \cap {}_{\mathcal{G}_\mathcal{N}}Cl(B) = \emptyset$.
2. Any two nonempty $\mathcal{G}_\mathcal{N}-$preclosed sets in X are $\mathcal{G}_\mathcal{N}-$separated if they are disjoint sets.

Definition 2.3. A grill topological space $(X, \tau, \mathcal{G})$ is called *a $\mathcal{G}_\mathcal{N}-$disconnected space* if it is the union of two $\mathcal{G}_\mathcal{N}-$separated sets. Otherwise A grill topological space $(X, \tau, \mathcal{G})$ is called *a $\mathcal{G}_\mathcal{N}-$connected space.*

Theorem 2.4. Any grill topological space $(X, \tau, \mathcal{G})$ with a finite set X is a $\mathcal{G}_{\mathcal{N}}-$disconnected space if X has more that one point.

Proof. The proof of the theorem is clear. □

Theorem 2.5. Every disconnected space is a $\mathcal{G}_{\mathcal{N}}-$disconnected space.

Proof. The proof of the theorem is clear since ${}_{\mathcal{G}_{\mathcal{N}}}Cl(A) \subset Cl(A)$. □

The converse of the above theorem need not be true.

Example 2.6. In the grill topological space $(\mathbb{R}, \tau, \mathcal{G})$, where $\tau = \{\emptyset, \mathbb{R}\}$ and $\mathcal{G} = \{\mathbb{R}\}$, is $\mathcal{G}_{\mathcal{N}}-$disconnected space but it is a connected space.

Theorem 2.7. A grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_{\mathcal{N}}-$disconnected space if and only if it is the union of two disjoint nonempty $\mathcal{G}_{\mathcal{N}}-$preopen sets.

Proof. Suppose that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_{\mathcal{N}}-$disconnected space. Then X is the union of two $\mathcal{G}_{\mathcal{N}}-$separated sets, that is, there are two nonempty subsets A and B of X such that

$$ {}_{\mathcal{G}_{\mathcal{N}}}Cl(A) \cap B = \emptyset,\ A \cap {}_{\mathcal{G}_{\mathcal{N}}}Cl(B) = \emptyset \text{ and } A \cup B = X. $$

Take

$$ G = X - {}_{\mathcal{G}_{\mathcal{N}}}Cl(A) \text{ and } H = X - {}_{\mathcal{G}_{\mathcal{N}}}Cl(B). $$

Then G and H are $\mathcal{G}_{\mathcal{N}}-$preopen sets. Since $B \neq \emptyset$ and ${}_{\mathcal{G}_{\mathcal{N}}}Cl(A) \cap B = \emptyset$, then $B \subseteq X - {}_{\mathcal{G}_{\mathcal{N}}}Cl(A)$, that is,

$$ G = X - {}_{\mathcal{G}_{\mathcal{N}}}Cl(A) \neq \emptyset. $$

Similar $H \neq \emptyset$. Since

$$ {}_{\mathcal{G}_{\mathcal{N}}}Cl(A) \cap B = \emptyset,\ A \cap {}_{\mathcal{G}_{\mathcal{N}}}Cl(B) = \emptyset \text{ and } A \cup B = X, $$

then

$$ X - (G \cap H) = (X - G) \cup (X - H) = [{}_{\mathcal{G}_{\mathcal{N}}}Cl(A)] \cup [{}_{\mathcal{G}_{\mathcal{N}}}Cl(B)] = X. $$

That is, $G \cap H = \emptyset$ and

$$ \begin{aligned} G \cup H &= (X - {}_{\mathcal{G}_{\mathcal{N}}}Cl(A)) \cap (X - {}_{\mathcal{G}_{\mathcal{N}}}Cl(B)) \\ &= X - ({}_{\mathcal{G}_{\mathcal{N}}}Cl(A) \cap {}_{\mathcal{G}_{\mathcal{N}}}Cl(B)) \\ &\subseteq X - ({}_{\mathcal{G}_{\mathcal{N}}}Cl(A) \cap B) \\ &= X - \emptyset = X \end{aligned} $$

.
Conversely, suppose that $(X, \tau, \mathcal{G})$ is the union of two disjoint nonempty $\mathcal{G}_{\mathcal{N}}-$preopen subsets, say G and H. Take

$$ A = X - G \text{ and } B = X - H. $$

Then A and B are $\mathcal{G}_{\mathcal{N}}-$preclosed sets, that is, ${}_{\mathcal{G}_{\mathcal{N}}}Cl(A) = A$ and ${}_{\mathcal{G}_{\mathcal{N}}}Cl(B) = B$. Since $H \neq \emptyset$ and $H \cap G = \emptyset$, then $H \subseteq X - G = A$, that is, $A \neq \emptyset$. Similar $B \neq \emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$, then

$$ \begin{aligned} {}_{\mathcal{G}_{\mathcal{N}}}Cl(A) \cap B &= A \cap B = (X - G) \cap (X - H) \\ &= X - (G \cup H) = X - X = \emptyset. \end{aligned} $$

Similar, $A \cap {}_{\mathcal{G}_{\mathcal{N}}}Cl(B) = \emptyset$. Note that

$$ A \cup B = (X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X. $$

That is, $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_{\mathcal{N}}-$disconnected space. □

Corollary 2.8. A grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_{\mathcal{N}}-$disconnected space if and only if it is the union of two disjoint nonempty $\mathcal{G}_{\mathcal{N}}-$preclosed subsets.

Proof. Suppose that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_{\mathcal{N}}-$disconnected space. Then by Theorem (2.7), $(X, \tau, \mathcal{G})$ is the union of two disjoint nonempty $\mathcal{G}_{\mathcal{N}}-$preopen subsets, say G and H. Then $X - G$ and $X - H$ are $\mathcal{G}_{\mathcal{N}}-$preclosed subsets. Since $G \neq \emptyset$, $H \neq \emptyset$ and $X = G \cup H$ then $X - G \neq \emptyset$, $X - H \neq \emptyset$ and

$$(X - G) \cap (X - H) = X - (G \cup H) = X - X = \emptyset.$$

Since $G \cap H = \emptyset$ then

$$(X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X.$$

Hence X is the union of two disjoint nonempty $\mathcal{G}_{\mathcal{N}}-$preclosed subsets.
Conversely, suppose that $(X, \tau, \mathcal{G})$ is the union of two disjoint nonempty $\mathcal{G}_{\mathcal{N}}-$preclosed subsets, say G and H. Take

$$A = X - G \text{ and } B = X - H.$$

Then A and B are $\mathcal{G}_{\mathcal{N}}-$preopen sets. Since $H \neq \emptyset$ and $H \cap G = \emptyset$, then $H \subseteq X - G = A$, that is, $A \neq \emptyset$. Similar $B \neq \emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$, then

$$\begin{aligned} A \cap B &= (X - G) \cap (X - H) \\ &= X - (G \cup H) \\ &= X - X = \emptyset. \end{aligned}$$

and

$$A \cup B = (X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X.$$

Therefore by Theorem (2.7), $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_{\mathcal{N}}-$disconnected space. □

The $\mathcal{G}-$preopen cover (resp. $\mathcal{G}_{\mathcal{N}}-$preopen cover) of a subset A of a grill topological space $(X, \tau, \mathcal{G})$ is a collection $\{G_\lambda : \lambda \in I\}$ of $\mathcal{G}-$preopen (resp. $\mathcal{G}_{\mathcal{N}}-$preopen) subsets of X such that $A \subseteq \cup_{\lambda \in I} G_\lambda$, where I is an index set. In particular for X if $X = \cup_{\lambda \in I} G_\lambda$.

3 $\mathcal{G}_{\mathcal{N}}-$Precompact Spaces

Definition 3.1. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. A is called a $\mathcal{G}_{\mathcal{N}}-$precompact set in a grill topological space $(X, \tau, \mathcal{G})$ if for every $\mathcal{G}_{\mathcal{N}}-$preopen cover $\{G_\lambda : \lambda \in I\}$ of A has finite $\mathcal{G}_{\mathcal{N}}-$preopen subcover $\{G_{\lambda_k} : k = 1, 2, ..., n\}$ of A such that $A \subseteq \cup_{k=1}^{n} G_{\lambda_k}$.
Similar, X is called a $\mathcal{G}_{\mathcal{N}}-$precompact space if $X = \cup_{k=1}^{n} G_{\lambda_k}$.

Theorem 3.2. Every $\mathcal{G}_{\mathcal{N}}-$precompact set is compact set.

Proof. The proof of the theorem is clear, since every open set is $\mathcal{G}_{\mathcal{N}}-$preopen set. □

The converse of the above theorem need not be true.

Example 3.3. In the grill topological space $(\mathbb{R}, \tau, \mathcal{G})$, where $\tau = \{\emptyset, \mathbb{R}\}$ and $\mathcal{G} = \{\mathbb{R}\}$, is not $\mathcal{G}_{\mathcal{N}}-$precompact space but it is a compact space.

Theorem 3.4. Let $(X, \tau, \mathcal{G})$ be grill topological space and $A \subseteq X$. The set A is a $\mathcal{G}_{\mathcal{N}}-$precompact set in a grill topological space $(X, \tau, \mathcal{G})$ if and only if for every $\mathcal{G}-$preopen cover of A has finite $\mathcal{G}-$preopen subcover.

Proof. Suppose that for every $\mathcal{G}-$preopen cover $\{G_\lambda : \lambda \in I\}$ of A has finite $\mathcal{G}-$preopen subcover. Let $\{G_\lambda : \lambda \in I\}$ be a $\mathcal{G}_\mathcal{N}-$preopen cover of A and $A \subseteq \cup_{\lambda\in I} G_\lambda$. Then for each $x \in A$, there is $\lambda_x \in I$ such that $x \in G_{\lambda_x}$. Since G_{λ_x} is $\mathcal{G}_\mathcal{N}-$preopen set, then there is $\mathcal{G}-$preopen set U_{λ_x} containing x such that $U_{\lambda_x} - G_{\lambda_x}$ is a finite set. Then the collection $\{U_{\lambda_x} : x \in A\}$ forms $\mathcal{G}-$preopen cover of A. Then by the hypothesis, this collection has finite $\mathcal{G}-$preopen subcover $\{U_{\lambda_{x_k}} : k = 1, 2, ..., n\}$ of A such that $A \subseteq \cup_{k=1}^n U_{\lambda_{x_k}}$. Note that

$$A \subseteq \cup_{k=1}^n [(U_{\lambda_{x_k}} - G_{\lambda_{x_k}}) \cup G_{\lambda_{x_k}}] = [\cup_{k=1}^n (U_{\lambda_{x_k}} - G_{\lambda_{x_k}})] \cup [\cup_{k=1}^n G_{\lambda_{x_k}}].$$

For each x_k, the set $U_{\lambda_{x_k}} - G_{\lambda_{x_k}}$ is a finite set and there is a finite subset $I(x_k)$ of I such that

$$(U_{\lambda_{x_k}} - G_{\lambda_{x_k}}) \subseteq \cup\{G_\lambda : \lambda \in I(x_k)\}.$$

Hence

$$A \subseteq [\cup_{k=1}^n (\cup\{G_\lambda : \lambda \in I(x_k)\}] \cup [\cup_{k=1}^n G_{\lambda_{x_k}}].$$

That is, A is a $\mathcal{G}_\mathcal{N}-$precompact set.
Conversely, it is clear since every $\mathcal{G}-$preopen set is $\mathcal{G}_\mathcal{N}-$preopen set. □

Corollary 3.5. The grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\mathcal{N}-$precompact space if and only if for every $\mathcal{G}-$preopen cover of X has finite $\mathcal{G}-$preopen subcover.

Corollary 3.6. Every strongly compact space is $\mathcal{G}_\mathcal{N}-$precompact space.

Proof. Let $\{G_\lambda : \lambda \in I\}$ be a $\mathcal{G}-$preopen cover of A be a grill topological space and $\{G_\lambda : \lambda \in I\}$ be a $\mathcal{G}-$preopen cover of X and $A \subseteq \cup_{\lambda\in I} G_\lambda$. Then by Theorem (1.15), $\{G_\lambda : \lambda \in I\}$ is a preopen cover of X. Since (X, τ) is strongly compact space, then $\{G_\lambda : \lambda \in I\}$ has finite $\mathcal{G}-$preopen subcover. Hence by Theorem (3.4), X is a $\mathcal{G}_\mathcal{N}-$precompact space. □

Theorem 3.7. Every $\mathcal{G}_\mathcal{N}-$preclosed subset of $\mathcal{G}_\mathcal{N}-$precompact space is $\mathcal{G}_\mathcal{N}-$precompact set.

Proof. Suppose that F is a $\mathcal{G}_\mathcal{N}-$preclosed subset of $\mathcal{G}_\mathcal{N}-$precompact space $(X, \tau, \mathcal{G})$. Let $\{V_\lambda : \lambda \in I\}$ be any $\mathcal{G}_\mathcal{N}-$preopen cover of F, where I is an index set. Since F is a $\mathcal{G}_\mathcal{N}-$preclosed set in $(X, \tau, \mathcal{G})$ then $X - F$ is a $\mathcal{G}_\mathcal{N}-$preopen in $(X, \tau, \mathcal{G})$. Then

$$\{(X - F), V_\lambda : \lambda \in I\}$$

is $\mathcal{G}_\mathcal{N}-$preopen cover of $(X, \tau, \mathcal{G})$. Since $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_\mathcal{N}-$precompact space then there is a finite subcover

$$\{(X - F), V_{\lambda k} : k = 1, 2, ..., m\}$$

such that

$$X = (X - F) \cup [\cup_{k=1}^m V_{\lambda k}].$$

Hence $F \subseteq \cup_{k=1}^m V_{\lambda k}$. Hence F is a $\mathcal{G}_\mathcal{N}-$precompact set in $(X, \tau, \mathcal{G})$. □

Theorem 3.8. Let $(X, \tau, \mathcal{G})$ be a grill topological space. Every proper $\mathcal{G}_\mathcal{N}-$preclosed subset of X is $\mathcal{G}_\mathcal{N}-$precompact set if and only if X is a $\mathcal{G}_\mathcal{N}-$precompact space.

Proof. Suppose that every proper $\mathcal{G}_\mathcal{N}-$preclosed subset of X is $\mathcal{G}_\mathcal{N}-$precompact set. Let $\{H_\lambda : \lambda \in I\}$ be any $\mathcal{G}-$preopen cover of X. Choose $\lambda_0 \in I$ such that H_{λ_0} is a proper subset of X. Then $\{H_\lambda : \lambda \in I - \{\lambda_0\}\}$ is $\mathcal{G}-$preopen cover of a $\mathcal{G}_\mathcal{N}-$preclosed set $X - H_{\lambda_0}$. By Theorem (3.7), $\{H_\lambda : \lambda \in I - \{\lambda_0\}\}$ is $\mathcal{G}_\mathcal{N}-$preopen cover of a $\mathcal{G}_\mathcal{N}-$preclosed set $X - H_{\lambda_0}$. Since $X - H_{\lambda_0}$ is $\mathcal{G}_\mathcal{N}-$precompact set in X then there is a finite subcover

$$\{H_{\lambda k} : k = 1, 2, ..., m\}$$

such that

$$X - H_{\lambda_0} \subseteq \cup_{k=1}^{m} H_{\lambda k}$$

This implies,

$$X \subseteq H_{\lambda_0} \cup [\cup_{k=1}^{m} H_{\lambda k}].$$

That is, X is a $\mathcal{G}_{\mathcal{N}}-$precompact space.
Conversely, by Theorem (3.7). □

4 $\mathcal{G}_{\mathcal{N}}-$Separation Axioms

Definition 4.1. A grill topological space $(X, \tau, \mathcal{G})$ is called:
1. $_{\mathcal{G}\mathcal{N}} - T_2$ *space* if for two points $x \neq y \in X$, there are two $\mathcal{G}_{\mathcal{N}}-$preopen set G and U in X such that $x \in G$ and $y \in U$ and $U \cap G = \emptyset$.
2. $\mathcal{G}_{\mathcal{N}}-$*regular space* if for each closed set F in $(X, \tau, \mathcal{G})$ and each $x \notin F$, there are two $\mathcal{G}_{\mathcal{N}}-$preopen sets G and U in $(X, \tau, \mathcal{G})$ such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. A grill topological space $(X, \tau, \mathcal{G})$ is called $_{\mathcal{G}\mathcal{N}} - T_3$ *space* if it is $\mathcal{G}_{\mathcal{N}}-$regular space and T_1 space.
3. $\mathcal{G}_{\mathcal{N}}-$*normal space* if for each two disjoint closed sets F and M in $(X, \tau, \mathcal{G})$, there are two $\mathcal{G}_{\mathcal{N}}-$preopen sets G and U in $(X, \tau, \mathcal{G})$ such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. A grill topological space $(X, \tau, \mathcal{G})$ is called $_{\mathcal{G}\mathcal{N}} - T_4$ *space* if it is $\mathcal{G}_{\mathcal{N}}-$normal space and T_1 space.

Theorem 4.2. If (X, τ) is T_i space then the grill topological space $(X, \tau, \mathcal{G})$ is a $_{\mathcal{G}\mathcal{N}} - T_i$ space for all $i = 2, 3, 4$.

Proof. The proof of the theorem is clear since every open set is $\mathcal{G}_{\mathcal{N}}-$preopen set. □

The converse of the last theorem need not be true.

Example 4.3. A grill topological space $(X, \tau, \mathcal{G})$ is a $_{\mathcal{G}\mathcal{N}} - T_i$ space but the space is not T_i space for all $i = 2, 3, 4$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$ and $\mathcal{G} = \{\{c\}, \{a, c\}, \{b, c\}, X\}$.

Example 4.4. A grill topological space $(X, \tau, \mathcal{G})$ is a $_{\mathcal{G}\mathcal{N}} - T_i$ space but the space is not T_i space for all $i = 2, 3$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathcal{G} = \{\{c\}, \{a, c\}, \{b, c\}, X\}$.

Theorem 4.5. Every $_{\mathcal{G}\mathcal{N}} - T_3$ space is a $_{\mathcal{G}\mathcal{N}} - T_2$ space.

Proof. Let be a $_{\mathcal{G}\mathcal{N}} - T_3$ space and $x \neq y \in X$ be any points in $(X, \tau, \mathcal{G})$. Since (X, τ) is a T_1 space then by Theorem (1.3), $\{x\}$ is a closed set in (X, τ) and $y \notin \{x\}$. Since $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_{\mathcal{N}}-$regular space then there are two $\mathcal{G}_{\mathcal{N}}-$preopen sets G and U in $(X, \tau, \mathcal{G})$ such that $x \in \{x\} \subseteq G$, $y \in U$ and $U \cap G = \emptyset$. Hence $(X, \tau, \mathcal{G})$ is a $_{\mathcal{G}\mathcal{N}} - T_2$ space. □

Theorem 4.6. Every $_{\mathcal{G}\mathcal{N}} - T_4$ space is a $_{\mathcal{G}\mathcal{N}} - T_3$ space.

Proof. Let be a $_{\mathcal{G}\mathcal{N}} - T_4$ space. Let F be any closed set in $(X, \tau, \mathcal{G})$ and $x \notin F$ be any point in $(X, \tau, \mathcal{G})$. Since (X, τ) is a T_1 space then by Theorem (1.3), $\{x\}$ is a closed set in (X, τ) and $F \cap \{x\} = \emptyset$. Since $(X, \tau, \mathcal{G})$ is a $\mathcal{G}_{\mathcal{N}}-$normal space then there are two $\mathcal{G}_{\mathcal{N}}-$preopen sets G and U in $(X, \tau, \mathcal{G})$ such that $x \in \{x\} \subseteq G$, $F \subseteq U$ and $U \cap G = \emptyset$. Hence $(X, \tau, \mathcal{G})$ is a $_{\mathcal{G}\mathcal{N}} - T_3$ space. □

Theorem 4.7. A grill topological space $(X, \tau, \mathcal{G})$ is $_{\mathcal{G}\mathcal{N}} - T_2$ space if and only if for each $x \in X$ and for $x \neq y \in X$, there is a $\mathcal{G}_{\mathcal{N}}-$preopen set H in $(X, \tau, \mathcal{G})$ containing x such that $y \notin {}_{\mathcal{G}\mathcal{N}}Cl(H)$.

Proof. Suppose that $(X, \tau, \mathcal{G})$ is ${}_{\mathcal{G}\mathcal{N}} - T_2$ space. Let $x \in X$ be any point in X and $x \neq y \in X$ be any point in X. Then there are two $\mathcal{G}_{\mathcal{N}}-$preopen sets G and U in $(X, \tau, \mathcal{G})$ such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$. Take $H = G$ is a $\mathcal{G}_{\mathcal{N}}-$preopen set in $(X, \tau, \mathcal{G})$ containing x and so $y \notin H \subseteq {}_{\mathcal{G}\mathcal{N}}Cl(H)$. Conversely, Let $x \neq y \in X$ be any points in $(X, \tau, \mathcal{G})$. and By the hypothesis, there is a $\mathcal{G}_{\mathcal{N}}-$preopen set H in $(X, \tau, \mathcal{G})$ containing x such that $y \notin {}_{\mathcal{G}\mathcal{N}}Cl(H)$. Then $X - {}_{\mathcal{G}\mathcal{N}}Cl(H)$ is a $\mathcal{G}_{\mathcal{N}}-$preopen sets in $(X, \tau, \mathcal{G})$ containing y such that $x \in H$, $y \in X - {}_{\mathcal{G}\mathcal{N}}Cl(H)$ and $H \cap X - {}_{\mathcal{G}\mathcal{N}}Cl(H) = \emptyset$. Then $(X, \tau, \mathcal{G})$ is ${}_{\mathcal{G}\mathcal{N}} - T_2$ space. □

Theorem 4.8. A grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{\mathcal{N}}-$regular space if and only if for each $x \in X$ and for each open set M in $(X, \tau, \mathcal{G})$ containing x, there is a $\mathcal{G}_{\mathcal{N}}-$preopen set H in $(X, \tau, \mathcal{G})$ containing x such that ${}_{\mathcal{G}\mathcal{N}}Cl(H) \subseteq M$.

Proof. Suppose that $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{\mathcal{N}}-$regular space. Let $x \in X$ be any point in X and M be any open set in $(X, \tau, \mathcal{G})$ containing x. Then $X - M$ is a closed set in $(X, \tau, \mathcal{G})$ and $x \notin X - M$. Since $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{\mathcal{N}}-$regular space then there are two $\mathcal{G}_{\mathcal{N}}-$preopen sets G and U in $(X, \tau, \mathcal{G})$ such that $(X - M) \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. Take $H = U$ is a $\mathcal{G}_{\mathcal{N}}-$preopen set in $(X, \tau, \mathcal{G})$ containing x. Then $H = U \subseteq (X - G)$, this implies,

$${}_{\mathcal{G}\mathcal{N}}Cl(H) \subseteq {}_{\mathcal{G}\mathcal{N}}Cl(X - G) \subseteq (X - G) \subseteq M.$$

Conversely, Let F be any closed set in $(X, \tau, \mathcal{G})$ and $x \notin F$. Then $x \in (X - F)$ and $X - F$ is an open set in $(X, \tau, \mathcal{G})$ containing x. By the hypothesis, there is a $\mathcal{G}_{\mathcal{N}}-$preopen set H in $(X, \tau, \mathcal{G})$ containing x such that ${}_{\mathcal{G}\mathcal{N}}Cl(H) \subseteq (X - F)$. Then $F \subseteq [X - {}_{\mathcal{G}\mathcal{N}}Cl(H)]$ and $X - {}_{\mathcal{G}\mathcal{N}}Cl(H)$ is a $\mathcal{G}_{\mathcal{N}}-$preopen set in $(X, \tau, \mathcal{G})$. Since H is a $\mathcal{G}_{\mathcal{N}}-$preopen set in $(X, \tau, \mathcal{G})$ containing x and $H \cap [X - {}_{\mathcal{G}\mathcal{N}}Cl(H)] = \emptyset$, then $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{\mathcal{N}}-$regular space. □

Theorem 4.9. A grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{\mathcal{N}}-$normal space if and only if for each closed set F in $(X, \tau, \mathcal{G})$ and for each open set G in $(X, \tau, \mathcal{G})$ containing F, there is a $\mathcal{G}_{\mathcal{N}}-$preopen set V in $(X, \tau, \mathcal{G})$ containing F such that ${}_{\mathcal{G}\mathcal{N}}Cl(V) \subseteq G$.

Proof. Suppose that $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{\mathcal{N}}-$normal space. Let F be any closed set in $(X, \tau, \mathcal{G})$ and G be any open set in $(X, \tau, \mathcal{G})$ containing F. Then $X - G$ is a closed set in $(X, \tau, \mathcal{G})$ and $F \cap (X - G) = \emptyset$. Since $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{\mathcal{N}}-$normal space then there are two $\mathcal{G}_{\mathcal{N}}-$preopen sets H and U in $(X, \tau, \mathcal{G})$ such that $(X - G) \subseteq U$, $F \subseteq H$ and $U \cap H = \emptyset$. Take $V = H$ is a $\mathcal{G}_{\mathcal{N}}-$preopen set in $(X, \tau, \mathcal{G})$ containing F. Then $V = H \subseteq (X - U)$, this implies,

$${}_{\mathcal{G}\mathcal{N}}Cl(V) \subseteq {}_{\mathcal{G}\mathcal{N}}Cl(X - U) \subseteq (X - U) \subseteq G.$$

Conversely, Let F and H be any two closed sets in $(X, \tau, \mathcal{G})$ such that $F \cap H = \emptyset$. Then $H \subseteq (X - F)$ and $X - F$ is an open set in $(X, \tau, \mathcal{G})$ containing closed set H. By the hypothesis, there is a $\mathcal{G}_{\mathcal{N}}-$preopen set V in $(X, \tau, \mathcal{G})$ containing H such that ${}_{\mathcal{G}\mathcal{N}}Cl(V) \subseteq (X - F)$. Then $F \subseteq X - {}_{\mathcal{G}\mathcal{N}}Cl(V)$ and $X - {}_{\mathcal{G}\mathcal{N}}Cl(V)$ is a $\mathcal{G}_{\mathcal{N}}-$preopen set in $(X, \tau, \mathcal{G})$. Since V is a $\mathcal{G}_{\mathcal{N}}-$preopen set in $(X, \tau, \mathcal{G})$ containing H and $V \cap [X - {}_{\mathcal{G}\mathcal{N}}Cl(V)] = \emptyset$, then $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{\mathcal{N}}-$normal space. □

5 Conclusion

The applications of $\mathcal{N}-$disconnected space, $\mathcal{N}-$precompact space, and separation axioms are well-known and important in the area of mathematics, computer science and other areas. The our notions in this article be development for the last notions in topological space into grill topological space by giving the concept is a strong, and will play the significant role in solving some mathematical problems.

Competing Interests

Authors have declared that no competing interests exist.

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