On Generalized Grahaml Numbers

Yüksel Soykan[1*]

[1]*Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100 Zonguldak, Turkey.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Original Research Article

Abstract

In this paper, we introduce the generalized Grahaml sequences and we deal with, in detail, three special cases which we call them Grahaml, Grahaml-Lucas and modified Grahaml sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

1 Introduction

In this paper, we investigate the generalized Grahaml sequences and we investigate, in detail, three special cases which we call them Grahaml, Grahaml-Lucas and modified Grahaml sequences.

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \quad F_0 = 0, \ F_1 = 1,$$

---

*Corresponding author: E-mail: yuksel_soykan@hotmail.com;

and

$$L_n = L_{n-1} + L_{n-2}, \;\; n \geq 2, \quad L_0 = 2, \; L_1 = 1$$

respectively. The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences.

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n \geq 0}$ (or shortly $\{W_n\}_{n \geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \;\; n \geq 3 \tag{1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [1,2,3,4,5,6,7,8,9,10,11,12,13].

The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, \dots$ when $t \neq 0$. Therefore, recurrence (1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{2}$$

whose roots are

$$\alpha = \alpha(r, s, t) = \frac{r}{3} + A + B$$
$$\beta = \beta(r, s, t) = \frac{r}{3} + \omega A + \omega^2 B$$
$$\gamma = \gamma(r, s, t) = \frac{r}{3} + \omega^2 A + \omega B$$

where

$$A = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \; B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3}$$
$$\Delta = \Delta(r, s, t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \;\; \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)$$

Note that we have the following identities

$$\alpha + \beta + \gamma = r,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -s,$$
$$\alpha\beta\gamma = t.$$

If $\Delta(r, s, t) > 0$, then the Equ. (2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{b_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{3}$$

where

$$b_1 = W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0, \; b_2 = W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0, \; b_3 = W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0.$$

Note that the Binet form of a sequence satisfying (2) for non-negative integers is valid for all integers n, for a proof of this result see [14]. This result of Howard and Saidak [14] is even true in the case of higher-order recurrence relations.

In this paper we consider the case $r = 2$, $s = 3, t = 5$ and in this case we write $V_n = W_n$. A generalized Grahaml sequence $\{V_n\}_{n \geq 0} = \{V_n(V_0, V_1, V_2)\}_{n \geq 0}$ is defined by the third-order recurrence relations

$$V_n = 2V_{n-1} + 3V_{n-2} + 5V_{n-3} \tag{4}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{3}{5}V_{-(n-1)} - \frac{2}{5}V_{-(n-2)} + \frac{1}{5}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (4) holds for all integer n.

(3) can be used to obtain Binet formula of generalized Grahaml numbers. Binet formula of generalized Grahaml numbers can be given as

$$V_n = \frac{b_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)}$$

where

$$b_1 = V_2 - (\beta + \gamma)V_1 + \beta\gamma V_0, \ b_2 = V_2 - (\alpha + \gamma)V_1 + \alpha\gamma V_0, \ b_3 = V_2 - (\alpha + \beta)V_1 + \alpha\beta V_0. \tag{5}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - 2x^2 - 3x - 5 = 0$. Moreover

$$\alpha = \frac{2}{3} + \left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3}$$

$$\beta = \frac{2}{3} + \omega\left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \omega^2\left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3}$$

$$\gamma = \frac{2}{3} + \omega^2\left(\frac{205}{54} + \sqrt{\frac{1231}{108}}\right)^{1/3} + \omega\left(\frac{205}{54} - \sqrt{\frac{1231}{108}}\right)^{1/3}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)$$

Note that

$$\alpha + \beta + \gamma = 2,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -3,$$
$$\alpha\beta\gamma = 5.$$

The first few generalized Grahaml numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Grahaml numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$\frac{1}{5}V_2 - \frac{2}{5}V_1 - \frac{3}{5}V_0$
2	V_2	$\frac{11}{25}V_1 - \frac{1}{25}V_0 - \frac{3}{25}V_2$
3	$5V_0 + 3V_1 + 2V_2$	$\frac{58}{125}V_0 - \frac{13}{125}V_1 - \frac{1}{125}V_2$
4	$10V_0 + 11V_1 + 7V_2$	$\frac{58}{625}V_2 - \frac{121}{625}V_1 - \frac{239}{625}V_0$
5	$35V_0 + 31V_1 + 25V_2$	$\frac{112}{3125}V_0 + \frac{768}{3125}V_1 - \frac{239}{3125}V_2$
6	$125V_0 + 110V_1 + 81V_2$	$\frac{3504}{15\,625}V_0 - \frac{1419}{15\,625}V_1 + \frac{112}{15\,625}V_2$
7	$405V_0 + 368V_1 + 272V_2$	$\frac{3504}{78\,125}V_2 - \frac{6448}{78\,125}V_1 - \frac{17\,607}{78\,125}V_0$
8	$1360V_0 + 1221V_1 + 912V_2$	$\frac{20\,581}{390\,625}V_0 + \frac{52\,734}{390\,625}V_1 - \frac{17\,607}{390\,625}V_2$

Now we define three special cases of the sequence $\{V_n\}$. Grahaml sequence$\{G_n\}_{n\geq 0}$, Grahaml-Lucas sequence $\{H_n\}_{n\geq 0}$ and modified Grahaml sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+3} = 2G_{n+2} + 3G_{n+1} + 5G_n, \qquad G_0 = 0, G_1 = 1, G_2 = 2,$$

$$H_{n+3} = 2H_{n+2} + 3H_{n+1} + 5H_n, \qquad H_0 = 3, H_1 = 2, H_2 = 10, \tag{6}$$

and

$$E_{n+3} = 2E_{n+2} + 3E_{n+1} + 5E_n, \qquad E_0 = 0, E_1 = 1, E_2 = 1, \tag{7}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{3}{5}G_{-(n-1)} - \frac{2}{5}G_{-(n-2)} + \frac{1}{5}G_{-(n-3)}, \tag{8}$$

$$H_{-n} = -\frac{3}{5}H_{-(n-1)} - \frac{2}{5}H_{-(n-2)} + \frac{1}{5}H_{-(n-3)} \tag{9}$$

and

$$E_{-n} = -\frac{3}{5}E_{-(n-1)} - \frac{2}{5}E_{-(n-2)} + \frac{1}{5}E_{-(n-3)} \tag{10}$$

for $n = 1, 2, 3, \dots$ respectively. Therefore, recurrences (8), (9) and (10) hold for all integer n.

Note that the sequences G_n, H_n and E_n are not indexed in [15] yet. Next, we present the first few values of the Grahaml, Grahaml-Lucas and modified Grahaml numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10
G_n	0	1	2	7	25	81	272	912	3045	10186	34067
G_{-n}		0	$\frac{1}{5}$	$-\frac{3}{25}$	$-\frac{1}{125}$	$\frac{58}{625}$	$-\frac{239}{3125}$	$\frac{112}{15625}$	$\frac{3504}{78125}$	$-\frac{17607}{390625}$	$\frac{20581}{1953125}$
H_n	3	2	10	41	122	417	1405	4671	15642	52322	174925
H_{-n}		$-\frac{3}{5}$	$-\frac{11}{25}$	$\frac{138}{125}$	$-\frac{379}{625}$	$-\frac{518}{3125}$	$\frac{8794}{15625}$	$-\frac{30677}{78125}$	$-\frac{8859}{390625}$	$\frac{553197}{1953125}$	$-\frac{2337926}{9765625}$
E_n	0	1	1	5	18	56	191	640	2133	7141	23881
E_{-n}		$-\frac{1}{5}$	$\frac{8}{25}$	$-\frac{14}{125}$	$-\frac{63}{625}$	$\frac{529}{3125}$	$-\frac{1307}{15625}$	$-\frac{2944}{78125}$	$\frac{35127}{390625}$	$-\frac{108616}{1953125}$	$-\frac{99022}{9765625}$

For all integers n, Grahaml, Grahaml-Lucas and modified Grahaml numbers (using initial conditions in (5)) can be expressed using Binet's formulas as

$$G_n = \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)},$$

and
$$H_n = \alpha^n + \beta^n + \gamma^n,$$

and
$$E_n = \frac{(\alpha - 1)\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{(\beta - 1)\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{(\gamma - 1)\gamma^n}{(\gamma - \alpha)(\gamma - \beta)},$$

respectively.

2 Generating Functions

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 2.1. Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Grahaml sequence $\{V_n\}_{n \geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2}{1 - 2x - 3x^2 - 5x^3}. \tag{11}$$

Proof. Using the definition of generalized Grahaml numbers, and substracting $2x \sum_{n=0}^{\infty} V_n x^n$, $3x^2 \sum_{n=0}^{\infty} V_n x^n$ and $5x^3 \sum_{n=0}^{\infty} V_n x^n$ from $\sum_{n=0}^{\infty} V_n x^n$ we obtain

$$
\begin{aligned}
(1 - 2x - 3x^2 - 5x^3) \sum_{n=0}^{\infty} V_n x^n &= \sum_{n=0}^{\infty} V_n x^n - 2x \sum_{n=0}^{\infty} V_n x^n - 3x^2 \sum_{n=0}^{\infty} V_n x^n - 5x^3 \sum_{n=0}^{\infty} V_n x^n \\
&= \sum_{n=0}^{\infty} V_n x^n - 2 \sum_{n=0}^{\infty} V_n x^{n+1} - 3 \sum_{n=0}^{\infty} V_n x^{n+2} - 5 \sum_{n=0}^{\infty} V_n x^{n+3} \\
&= \sum_{n=0}^{\infty} V_n x^n - 2 \sum_{n=1}^{\infty} V_{n-1} x^n - 3 \sum_{n=2}^{\infty} V_{n-2} x^n - 5 \sum_{n=3}^{\infty} V_{n-3} x^n \\
&= (V_0 + V_1 x + V_2 x^2) - 2(V_0 x + V_1 x^2) - 3V_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (V_n - 2V_{n-1} - 3V_{n-2} - 5V_{n-3}) x^n \\
&= V_0 + V_1 x + V_2 x^2 - 2V_0 x - 2V_1 x^2 - 3V_0 x^2 \\
&= V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2.
\end{aligned}
$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2}{1 - 2x - 3x^2 - 5x^3}.$$

The previous Lemma gives the following results as particular examples.

Corollary 2.1. *Generated functions of Grahaml, Grahaml-Lucas and modified Grahaml numbers are*

$$\sum_{n=0}^{\infty} G_n x^n = \frac{x}{1 - 2x - 3x^2 - 5x^3},$$

and

$$\sum_{n=0}^{\infty} H_n x^n = \frac{3 - 4x - 3x^2}{1 - 2x - 3x^2 - 5x^3},$$

and

$$\sum_{n=0}^{\infty} E_n x^n = \frac{x - x^2}{1 - 2x - 3x^2 - 5x^3},$$

respectively.

3   Obtaining Binet Formula from Generating Function

We next find Binet formula of generalized Grahaml numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 3.1. *(Binet formula of generalized Grahaml numbers)*

$$V_n = \frac{d_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{d_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{d_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{12}$$

where

$$\begin{aligned}
d_1 &= V_0 \alpha^2 + (V_1 - 2V_0)\alpha + (V_2 - 2V_1 - 3V_0), \\
d_2 &= V_0 \beta^2 + (V_1 - 2V_0)\beta + (V_2 - 2V_1 - 3V_0), \\
d_3 &= V_0 \gamma^2 + (V_1 - 2V_0)\gamma + (V_2 - 2V_1 - 3V_0).
\end{aligned}$$

Proof. Let

$$h(x) = 1 - 2x - 3x^2 - 5x^3.$$

Then for some α, β and γ we write

$$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)$$

i.e.,

$$1 - 2x - 3x^2 - 5x^3 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x) \tag{13}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ are the roots of $h(x)$. This gives α, β, and γ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{2}{x} - \frac{3}{x^2} - \frac{5}{x^3} = 0.$$

This implies $x^3 - 2x^2 - 3x - 5 = 0$. Now, by (11) and (13), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)}.$$

Then we write

$$\frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)} = \frac{A_1}{(1 - \alpha x)} + \frac{A_2}{(1 - \beta x)} + \frac{A_3}{(1 - \gamma x)}. \tag{14}$$

So

$$V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2 = A_1(1 - \beta x)(1 - \gamma x) + A_2(1 - \alpha x)(1 - \gamma x) + A_3(1 - \alpha x)(1 - \beta x).$$

If we consider $x = \frac{1}{\alpha}$, we get $V_0 + (V_1 - 2V_0)\frac{1}{\alpha} + (V_2 - 2V_1 - 3V_0)\frac{1}{\alpha^2} = A_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})$. This gives

$$A_1 = \frac{\alpha^2(V_0 + (V_1 - 2V_0)\frac{1}{\alpha} + (V_2 - 2V_1 - 3V_0)\frac{1}{\alpha^2})}{(\alpha - \beta)(\alpha - \gamma)} = \frac{V_0 \alpha^2 + (V_1 - 2V_0)\alpha + (V_2 - 2V_1 - 3V_0)}{(\alpha - \beta)(\alpha - \gamma)}.$$

Similarly, we obtain

$$A_2 = \frac{V_0 \beta^2 + (V_1 - 2V_0)\beta + (V_2 - 2V_1 - 3V_0)}{(\beta - \alpha)(\beta - \gamma)}, A_3 = \frac{V_0 \gamma^2 + (V_1 - 2V_0)\gamma + (V_2 - 2V_1 - 3V_0)}{(\gamma - \alpha)(\gamma - \beta)}.$$

Thus (14) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1 (1 - \alpha x)^{-1} + A_2 (1 - \beta x)^{-1} + A_3 (1 - \gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} V_n x^n = A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n = \sum_{n=0}^{\infty} (A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n) x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n$$

where

$$A_1 = \frac{V_0 \alpha^2 + (V_1 - 2V_0)\alpha + (V_2 - 2V_1 - 3V_0)}{(\alpha - \beta)(\alpha - \gamma)},$$

$$A_2 = \frac{V_0 \beta^2 + (V_1 - 2V_0)\beta + (V_2 - 2V_1 - 3V_0)}{(\beta - \alpha)(\beta - \gamma)},$$

$$A_3 = \frac{V_0 \gamma^2 + (V_1 - 2V_0)\gamma + (V_2 - 2V_1 - 3V_0)}{(\gamma - \alpha)(\gamma - \beta)},$$

and then we get (12).

Note that from (5) and (12) we have

$$\begin{aligned}
V_2 - (\beta + \gamma)V_1 + \beta\gamma V_0 &= V_0 \alpha^2 + (V_1 - 2V_0)\alpha + (V_2 - 2V_1 - 3V_0), \\
V_2 - (\alpha + \gamma)V_1 + \alpha\gamma V_0 &= V_0 \beta^2 + (V_1 - 2V_0)\beta + (V_2 - 2V_1 - 3V_0), \\
V_2 - (\alpha + \beta)V_1 + \alpha\beta V_0 &= V_0 \gamma^2 + (V_1 - 2V_0)\gamma + (V_2 - 2V_1 - 3V_0).
\end{aligned}$$

Next, using Theorem 3.1, we present the Binet formulas of Grahaml, Grahaml-Lucas and modified Grahaml sequences.

Corollary 3.2. *Binet formulas of Grahaml, Grahaml-Lucas and modified Grahaml sequences are*

$$G_n = \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)},$$

and

$$H_n = \alpha^n + \beta^n + \gamma^n,$$

and

$$E_n = \frac{(\alpha - 1)\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{(\beta - 1)\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{(\gamma - 1)\gamma^n}{(\gamma - \alpha)(\gamma - \beta)},$$

respectively.

We can find Binet formulas by using matrix method with a similar technique which is given in [16]. Take $k = i = 3$ in Corollary 3.1 in [16]. Let

$$\Lambda = \begin{pmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{pmatrix},$$

$$\Lambda_2 = \begin{pmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{pmatrix}, \Lambda_3 = \begin{pmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{pmatrix}.$$

Then the Binet formula for Grahaml numbers is

$$
\begin{aligned}
G_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{3} G_{4-j} \det(\Lambda_j) = \frac{1}{\Lambda}(G_3 \det(\Lambda_1) + G_2 \det(\Lambda_2) + G_1 \det(\Lambda_3)) \\
&= \frac{1}{\det(\Lambda)}(7 \det(\Lambda_1) + 2 \det(\Lambda_2) + \det(\Lambda_3)) \\
&= \left(7 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 2 \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) \Big/ \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}.
\end{aligned}
$$

Similarly, we obtain the Binet formula for Grahaml-Lucas and modified Grahaml numbers as

$$
\begin{aligned}
H_n &= \frac{1}{\Lambda}(H_3 \det(\Lambda_1) + H_2 \det(\Lambda_2) + H_1 \det(\Lambda_3)) \\
&= \left(41 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 10 \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + 2 \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) \Big/ \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}.
\end{aligned}
$$

and

$$
\begin{aligned}
E_n &= \frac{1}{\Lambda}(E_3 \det(\Lambda_1) + E_2 \det(\Lambda_2) + E_1 \det(\Lambda_3)) \\
&= \left(5 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) \Big/ \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}.
\end{aligned}
$$

respectively.

4   Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$
F_{n+1} F_{n-1} - F_n^2 = (-1)^n
$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$
\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.
$$

The following theorem gives generalization of this result to the generalized Grahaml sequence $\{V_n\}_{n \geq 0}$.

Theorem 4.1 (Simson Formula of Generalized Grahaml Numbers)**.** *For all integers n, we have*

$$
\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = 5^n \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{15}
$$

Proof. (15) is given in Soykan [17].

The previous theorem gives the following results as particular examples.

Corollary 4.2. *For all integers n, Simson formula of Grahaml, Grahaml-Lucas and modified Grahaml numbers are given as*

$$
\begin{vmatrix}
G_{n+2} & G_{n+1} & G_n \\
G_{n+1} & G_n & G_{n-1} \\
G_n & G_{n-1} & G_{n-2}
\end{vmatrix} = -5^{n-1},
$$

and

$$
\begin{vmatrix}
H_{n+2} & H_{n+1} & H_n \\
H_{n+1} & H_n & H_{n-1} \\
H_n & H_{n-1} & H_{n-2}
\end{vmatrix} = -1231 \times 5^{n-2},
$$

and

$$
\begin{vmatrix}
E_{n+2} & E_{n+1} & E_n \\
E_{n+1} & E_n & E_{n-1} \\
E_n & E_{n-1} & E_{n-2}
\end{vmatrix} = -9 \times 5^{n-2},
$$

respectively.

5 Some Identities

In this section, we obtain some identities of Grahaml, Grahaml-Lucas and modified Grahaml numbers. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 5.1. The following equalities are true:

$$
\begin{aligned}
125H_n &= 138G_{n+4} - 331G_{n+3} - 379G_{n+2}, & (16) \\
25H_n &= -11G_{n+3} + 7G_{n+2} + 138G_{n+1}, \\
5H_n &= -3G_{n+2} + 21G_{n+1} - 11G_n, \\
H_n &= 3G_{n+1} - 4G_n - 3G_{n-1}, \\
H_n &= 2G_n + 6G_{n-1} + 15G_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
6155G_n &= 18H_{n+4} - 166H_{n+3} + 461H_{n+2}, \\
1231G_n &= -26H_{n+3} + 103H_{n+2} + 18H_{n+1}, \\
1231G_n &= 51H_{n+2} - 60H_{n+1} - 130H_n, \\
1231G_n &= 42H_{n+1} + 23H_n + 255H_{n-1}, \\
1231G_n &= 107H_n + 381H_{n-1} + 210H_{n-2}.
\end{aligned}
$$

Proof. Note that all the identities hold for all integers n. We prove (16). To show (16), writing

$$
H_n = a \times G_{n+4} + b \times G_{n+3} + c \times G_{n+2}
$$

and solving the system of equations

$$
\begin{aligned}
H_0 &= a \times G_4 + b \times G_3 + c \times G_2 \\
H_1 &= a \times G_5 + b \times G_4 + c \times G_3 \\
H_2 &= a \times G_6 + b \times G_5 + c \times G_4
\end{aligned}
$$

we find that $a = \frac{138}{125}, b = -\frac{331}{125}, c = -\frac{379}{125}$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

Lemma 5.2. The following equalities are true:

$$
\begin{aligned}
125E_n &= -14G_{n+4} + 68G_{n+3} - 63G_{n+2}, \\
25E_n &= 8G_{n+3} - 21G_{n+2} - 14G_{n+1}, \\
5E_n &= -G_{n+2} + 2G_{n+1} + 8G_n, \\
E_n &= G_n - G_{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
45G_n &= -4E_{n+4} + 13E_{n+3} + 7E_{n+2}, \\
9G_n &= E_{n+3} - E_{n+2} - 4E_{n+1}, \\
9G_n &= E_{n+2} - E_{n+1} + 5E_n, \\
9G_n &= E_{n+1} + 8E_n + 5E_{n-1}, \\
9G_n &= 10E_n + 8E_{n-1} + 5E_{n-2}.
\end{aligned}
$$

Thirdly, we give a few basic relations between $\{H_n\}$ and $\{E_n\}$.

Lemma 5.3. The following equalities are true:

$$
\begin{aligned}
225H_n &= 134E_{n+4} - 233E_{n+3} - 572E_{n+2}, \\
45H_n &= 7E_{n+3} - 34E_{n+2} + 134E_{n+1}, \\
9H_n &= -4E_{n+2} + 31E_{n+1} + 7E_n, \\
9H_n &= 23E_{n+1} - 5E_n - 20E_{n-1}, \\
9H_n &= 41E_n + 49E_{n-1} + 115E_{n-2}.
\end{aligned}
$$

and

$$
\begin{aligned}
30\,775E_n &= -371H_{n+4} + 2H_{n+3} + 4518H_{n+2}, \\
6155E_n &= -148H_{n+3} + 681H_{n+2} - 371H_{n+1}, \\
1231E_n &= 77H_{n+2} - 163H_{n+1} - 148H_n, \\
1231E_n &= -9H_{n+1} + 83H_n + 385H_{n-1}, \\
1231E_n &= 65H_n + 358H_{n-1} - 45H_{n-2}.
\end{aligned}
$$

We now present a few special identities for the modified Grahaml sequence $\{E_n\}$.

Theorem 5.1. *(Catalan's identity) For all integers n and m, the following identity holds*

$$
\begin{aligned}
E_{n+m}E_{n-m} - E_n^2 &= (G_{n+m} - G_{n+m-1})(G_{n-m} - G_{n-m-1}) - (G_n - G_{n-1})^2 \\
&= (G_n(G_m - G_{m+1}) + G_{n-1}(-G_m + G_{m-2}) + G_{n-2}(-G_m + G_{m-1})) \\
&\quad (G_n(G_{-m} - G_{1-m}) + G_{n-1}(-G_{-m} + G_{-m-2}) + G_{n-2}(-G_{-m} + G_{-m-1})) \\
&\quad -(G_n - G_{n-1})^2
\end{aligned}
$$

Proof. We use the identity

$$
E_n = G_n - G_{n-1}
$$

and the identity (22).

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified Grahaml sequnce

Corollary 5.2. *(Cassini's identity) For all integers numbers n and m, the following identity holds*

$$E_{n+1}E_{n-1} - E_n^2 = (G_{n+1} - G_n)(G_{n-1} - G_{n-2}) - (G_n - G_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $E_n = G_n - G_{n-1}$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified Grahaml sequence $\{E_n\}$.

Theorem 5.3. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_m E_{n+1} = (G_{m+1} - G_m)(G_n - G_{n-1}) - (G_m - G_{m-1})(G_{n+1} - G_n).$$

(b) *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (G_{n+2}-G_{n+1})(G_{n+1}-G_n)(G_{n-1}-G_{n-2})(G_{n-2}-G_{n-3})-(G_n-G_{n-1})^4.$$

(c) *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (G_{n+1} - G_n)(G_{n+2} - G_{n+1})(G_{n+6} - G_{n+5}) - (G_{n+3} - G_{n+2})^3.$$

Proof. Use the identity $E_n = G_n - G_{n-1}$.

6 Linear Sums

The following proposition presents some formulas of generalized Grahaml numbers with positive subscripts.

Proposition 6.1. *If $r = 2, s = 3, t = 5$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^n V_k = \frac{1}{9}(V_{n+3} - V_{n+2} - 4V_{n+1} - V_2 + V_1 + 4V_0).$

(b) $\sum_{k=0}^n V_{2k} = \frac{1}{45}(-2V_{2n+2} + 11V_{2n+1} + 35V_{2n} + 2V_2 - 11V_1 + 10V_0).$

(c) $\sum_{k=0}^n V_{2k+1} = \frac{1}{45}(7V_{2n+2} + 29V_{2n+1} - 10V_{2n} - 7V_2 + 16V_1 + 10V_0).$

Proof. Take $r = 2, s = 3, t = 5$ in Theorem 2.1 in [18], see also [19].

As special cases of above proposition, we have the following three corollaries. First one presents some summing formulas of Grahaml numbers (take $V_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 6.1. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^n G_k = \frac{1}{9}(G_{n+3} - G_{n+2} - 4G_{n+1} - 1).$

(b) $\sum_{k=0}^n G_{2k} = \frac{1}{45}(-2G_{2n+2} + 11G_{2n+1} + 35G_{2n} - 7).$

(c) $\sum_{k=0}^n G_{2k+1} = \frac{1}{45}(7G_{2n+2} + 29G_{2n+1} - 10G_{2n} + 2).$

Second one presents some summing formulas of Grahaml-Lucas numbers (take $V_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$).

Corollary 6.2. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^n H_k = \frac{1}{9}(H_{n+3} - H_{n+2} - 4H_{n+1} + 4).$

(b) $\sum_{k=0}^{n} H_{2k} = \frac{1}{45}(-2H_{2n+2} + 11H_{2n+1} + 35H_{2n} + 28)$.

(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{45}(7H_{2n+2} + 29H_{2n+1} - 10H_{2n} - 8)$.

Third one presents some summing formulas of modified Grahaml numbers (take $V_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 6.3. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} E_k = \frac{1}{9}(E_{n+3} - E_{n+2} - 4E_{n+1})$.

(b) $\sum_{k=0}^{n} E_{2k} = \frac{1}{45}(-2E_{2n+2} + 11E_{2n+1} + 35E_{2n} - 9)$.

(c) $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{45}(7E_{2n+2} + 29E_{2n+1} - 10E_{2n} + 9)$.

The following proposition presents some formulas of generalized Grahaml numbers with negative subscripts.

Proposition 6.2. *If $r = 2, s = 3, t = 5$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} V_{-k} = \frac{1}{9}(-10V_{-n-1} - 8V_{-n-2} - 5V_{-n-3} + V_2 - V_1 - 4V_0)$.

(b) $\sum_{k=1}^{n} V_{-2k} = \frac{1}{45}(-7V_{-2n+1} + 16V_{-2n} + 10V_{-2n-1} - 2V_2 + 11V_1 - 10V_0)$.

(c) $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{45}(2V_{-2n+1} - 11V_{-2n} - 35V_{-2n-1} + 7V_2 - 16V_1 - 10V_0)$.

Proof. Take $r = 2, s = 3, t = 5$ in Theorem 3.1 in [18], see also [19].

From the above proposition, we have the following corollary which gives sum formulas of Grahaml numbers (take $V_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 6.4. *For $n \geq 1$, Grahaml numbers have the following properties.*

(a) $\sum_{k=1}^{n} G_{-k} = \frac{1}{9}(-10G_{-n-1} - 8G_{-n-2} - 5G_{-n-3} + 1)$.

(b) $\sum_{k=1}^{n} G_{-2k} = \frac{1}{45}(-7G_{-2n+1} + 16G_{-2n} + 10G_{-2n-1} + 7)$.

(c) $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{45}(2G_{-2n+1} - 11G_{-2n} - 35G_{-2n-1} - 2)$.

Taking $V_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last proposition, we have the following corollary which presents sum formulas of Grahaml -Lucas numbers.

Corollary 6.5. *For $n \geq 1$, Grahaml -Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} H_{-k} = \frac{1}{9}(-10H_{-n-1} - 8H_{-n-2} - 5H_{-n-3} - 4)$.

(b) $\sum_{k=1}^{n} H_{-2k} = \frac{1}{45}(-7H_{-2n+1} + 16H_{-2n} + 10H_{-2n-1} - 28)$.

(c) $\sum_{k=1}^{n} H_{-2k+1} = \frac{1}{45}(2H_{-2n+1} - 11H_{-2n} - 35H_{-2n-1} + 8)$.

From the above proposition, we have the following corollary which gives sum formulas of modified Grahaml numbers (take $V_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 6.6. *For $n \geq 1$, modified Grahaml numbers have the following properties.*

(a) $\sum_{k=1}^{n} E_{-k} = \frac{1}{9}(-10E_{-n-1} - 8E_{-n-2} - 5E_{-n-3})$.

(b) $\sum_{k=1}^{n} E_{-2k} = \frac{1}{45}(-7E_{-2n+1} + 16E_{-2n} + 10E_{-2n-1} + 9)$.

(c) $\sum_{k=1}^{n} E_{-2k+1} = \frac{1}{45}(2E_{-2n+1} - 11E_{-2n} - 35E_{-2n-1} - 9)$.

7 Matrices Related with Generalized Grahaml Numbers

Matrix formulation of W_n can be given as

$$
\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{17}
$$

For matrix formulation (17), see [20]. In fact, Kalman gave the formula in the following form

$$
\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.
$$

We define the square matrix A of order 3 as:

$$
A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
$$

such that $\det A = 5$. From (4) we have

$$
\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{18}
$$

and from (17) (or using (18) and induction) we have

$$
\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.
$$

If we take $V = G$ in (18) we have

$$
\begin{pmatrix} G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{19}
$$

We also define

$$
B_n = \begin{pmatrix} G_{n+1} & 3G_n + 5G_{n-1} & 5G_n \\ G_n & 3G_{n-1} + 5G_{n-2} & 5G_{n-1} \\ G_{n-1} & 3G_{n-2} + 5G_{n-3} & 5G_{n-2} \end{pmatrix}
$$

and

$$
C_n = \begin{pmatrix} V_{n+1} & 3V_n + 5V_{n-1} & 5V_n \\ V_n & 3V_{n-1} + 5V_{n-2} & 5V_{n-1} \\ V_{n-1} & 3V_{n-2} + 5V_{n-3} & 5V_{n-2} \end{pmatrix}
$$

Theorem 7.1. *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (19) to 3-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$AC_{n-1} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & 3V_{n-1}+5V_{n-2} & 5V_{n-1} \\ V_{n-1} & 3V_{n-2}+5V_{n-3} & 5V_{n-2} \\ V_{n-2} & 3V_{n-3}+5V_{n-4} & 5V_{n-3} \end{pmatrix}$$

$$= \begin{pmatrix} V_{n+1} & 3V_n+5V_{n-1} & 5V_n \\ V_n & 3V_{n-1}+5V_{n-2} & 5V_{n-1} \\ V_{n-1} & 3V_{n-2}+5V_{n-3} & 5V_{n-2} \end{pmatrix} = C_n.$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of matrix A^n can be given as

$$A^n = 2A^{n-1} + 3A^{n-2} + 5A^{n-3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 5^n$$

for all integer m and n.

Theorem 7.2. *For $m, n \geq 0$ we have*

$$
\begin{aligned}
V_{n+m} &= V_n G_{m+1} + V_{n-1}(3G_m + 5G_{m-1}) + 5V_{n-2}G_m & (20) \\
&= V_n G_{m+1} + (3V_{n-1} + 5V_{n-2})G_m + 5V_{n-1}G_{m-1} & (21)
\end{aligned}
$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof.

Remark 7.1. By induction, it can be proved that for all integers $m, n \leq 0$, (20) holds. So for all integers m, n, (20) is true.

Corollary 7.3. *For all integers m, n, we have*

$$
\begin{aligned}
G_{n+m} &= G_n G_{m+1} + G_{n-1}(3G_m + 5G_{m-1}) + 5G_{n-2}G_m, & (22) \\
H_{n+m} &= H_n G_{m+1} + H_{n-1}(3G_m + 5G_{m-1}) + 5H_{n-2}G_m, & (23) \\
E_{n+m} &= E_n G_{m+1} + E_{n-1}(3G_m + 5G_{m-1}) + 5E_{n-2}G_m. & (24)
\end{aligned}
$$

8 Conclusions

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. We introduce the generalized Grahaml sequence (it's three special cases, namely, Grahaml, Grahaml-Lucas and modified Grahaml sequences) and we present Binet's formulas, generating functions, Simson formulas, the summation formulas, some identities and matrices for these sequences. Generalized Grahaml sequence (and it's three special cases: Grahaml, Grahaml-Lucas and modified Grahaml sequences) can also be called (named) as generalized 3-primes sequence (3-primes, Lucas 3-primes and modified 3-primes sequences, respectively).

Competing Interests

Author has declared that no competing interests exist.

References

[1] Bruce I. A modified Tribonacci Sequence. Fibonacci Quarterly. 1984;22(3):244-246.

[2] Catalani M. Identities for Tribonacci-related sequences, arXiv:math/0209179; 2012.

[3] Choi E. Modular Tribonacci numbers by matrix method. Journal of the Korean Society of Mathematical Education Series B: Pure and Applied Mathematics. 2013;20(3):207-221.

[4] Elia M. Derived sequences. The Tribonacci Recurrence and Cubic Forms, Fibonacci Quarterly. 2001;39(2):107-115.

[5] Er MC. Sums of fibonacci numbers by matrix methods. Fibonacci Quarterly. 1984;22(3):204-207.

[6] Lin PY. De Moivre-type identities for the Tribonacci numbers. Fibonacci Quarterly. 1988;26:131-134.

[7] Pethe S. Some identities for Tribonacci sequences. Fibonacci Quarterly. 1988;26(2):144-151.

[8] Scott A, Delaney T, Hoggatt Jr. V. The Tribonacci sequence. Fibonacci Quarterly. 1977;15(3):193-200.

[9] Shannon A. Tribonacci numbers and Pascal's pyramid. Fibonacci Quarterly. 1977;15(3):268-275.

[10] Soykan Y. Tribonacci and Tribonacci-Lucas Sedenions. Mathematics. 2019;7(1):74.

[11] Spickerman W. Binet's formula for the Tribonacci sequence. Fibonacci Quarterly. 1982;20:118-120.

[12] Yalavigi CC. Properties of Tribonacci numbers. Fibonacci Quarterly. 1972;10(3):231-246.

[13] Yilmaz N, Taskara N. Tribonacci and Tribonacci-Lucas numbers via the determinants of special Matrices. Applied Mathematical Sciences. 2014;8(39):1947-1955.

[14] Howard FT, Saidak F. Zhou's theory of constructing identities. Congress Numer. 2010;200:225-237.

[15] Sloane NJA. The on-line encyclopedia of integer sequences.
Available:http://oeis.org/

[16] Kiliç E, Stanica P. A matrix approach for general higher order linear recurrences. Bulletin of the Malaysian Mathematical Sciences Society. 2011;34(1):51-67.

[17] Soykan Y. Simson identity of generalized m-step Fibonacci numbers. Int. J. Adv. Appl. Math. and Mech. 2019;7(2):45-56.

[18] Soykan Y. Summing formulas for generalized Tribonacci numbers. arXiv:1910.03490v1 [math.GM]; 2019.

[19] Soykan Y. Summing formulas for generalized Tribonacci numbers. Universal Journal of Mathematics and Applications, Accepted.

[20] Kalman D. Generalized Fibonacci numbers by matrix methods. Fibonacci Quarterly. 1982;20(1):73-76.