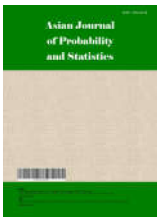

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Complex Dynamics in a Mixed Duopoly Game Based on Relative Profit Maximization

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Authors' contributions

This work was carried out in collaboration between both authors. Author YD designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author XL managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

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Abstract

In this paper, the complex dynamic behavior of a mixed duopoly game model is studied. Based on the principle of relative profit maximization and bounded rational expectation, the corresponding discrete dynamic systems are constructed in the case of nonlinear cost function. In theory, the conditions for the local stability of Nash equilibrium are given. In terms of numerical experiments, bifurcation diagrams are used to depict the effects of product differences, adjustment speed, and other parameters on the stability of Nash equilibrium.

Keywords: Mixed duopoly; relative profit maximization; bounded rational; nash equilibrium point; stability.

1 Introduction

Oligopoly market refers to a market with a limited number of enterprises at the same time, and different market players compete incompletely in the same market. In the market economy, the product price occupies a larger weight in the process of consumer choice, so the law of price competition is essential for enterprise

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market analysis. At the same time, the number of products has a profound impact on the market law, and it is also an important part of the market price competition. Therefore, it is particularly important to study the law of oligopoly game between product price and product quantity.

In recent years, many literatures in the world made a lot of analysis on the oligopoly game. Based on Cournot or Bertrand game in oligopolistic market [1-3], have studied a lot and reported a large amount of works. With the spillover effect, Zha et al. [4] studied the Nash equilibrium stability of Bertrand duopoly game. In [5], the dynamic characteristics of closed-loop Bertrand game have been investigated in supply chain. In [6], they analyze the strategic authorization model of Cournot competition under the condition of equal elastic demand. Reference [7] analyzes the heterogeneous game with linear demand function and parabolic total cost function. In [8], they consider a mixed Cournot Bertrand duopoly game, in which there are different types of competition. Chen et al. [9] introduced the Bertrand game with linear demand to simulate the competition in China's telecom market. Reference [10] studied a linear continuous Bertrand duopoly game model with time delay. In recent years, studies show that under conditions such as [11], the oligarchic market may have chaotic or cyclic behavior. In these studies, the main focus is to study the complex dynamic characteristics of the game. For example, some complex dynamics of duopoly game are studied in [12]. Askar and Al-khedhairi [12] used the nonlinear difference equation to establish a dynamic Cournot three-oligopoly game model, which consisted of three homogeneous bounded rational players. Andaluz et al. [13] have considered a Cournot oligopoly model which consists of three competitive companies that offer homogeneous goods. In [14], the dynamics of an oligopoly model considering various amendments and more firms are studied with various efforts. Reference [15] includes many in-depth studies on Cournot duopoly games, which have provided important results on the complex dynamics of such games. Using the technical innovation of naive expectations and bounded rationality, the complex dynamics of a Bertrand duopoly game in [16] have been studied.

The above literatures assume that the two competing firms adopt the same competitive strategies, such as output competition and price competition. However, the Cournot–Bertrand mixed competition, in which one firm adopts the price strategy and the other firm adopts the output strategy, widely exists in the actual industry. It is an intermediate structure between the complete output competition and the price competition. As stated by Tremblay and Tremblay [17], a growing body of work demonstrates that the Cournot–Bertrand outcome can be a subgameperfect Nash equilibrium in the presence of market asymmetries. Observations of real-world markets consistent with Cournot–Bertrand behavior bolster justification for the model and have stimulated an impressive and evolving literature on advances and applications. There are few works on discussing such types of games in literature, see [17] and the references therein for more details. Among these works, Tremblay et al. [18] have investigated the conditions of the stability for Nash equilibrium in the static case. Naimzada et al. [19]have adopted the best response mechanism with the adaptive adjustment approach to model the game and to study the stability of its equilibrium points. In the above mentioned works, the authors have used different kinds of adjustment mechanisms to model the suggested games, in which the bounded rationality seems the most popular one. Information about this mechanism and its analytical form can be founded in literature [20–21]. Recently, Askar [22] invested the complex dynamics of Cournot-Bertrand game with asymmetric market information, in which players adopt the bounded rationality approach and one player has some asymmetric information about the production of his opponent. This work gives a rich local and global analysis of the game’s dynamics that include multiple stable attractors and analyzing the basins of attraction for some attracting sets.

Inspired by the above studies, we consider the Complex dynamics in a mixed duopoly game based on relative profit maximization. In the case of nonlinear cost function, we construct the corresponding discrete dynamic system and study the stability of Nash equilibrium to provide theoretical support for competitive strategy in dynamic market competition.

The rest of this paper is arranged as follows: Section 2 gives the basic structure of the model; Section 3 studies the dynamic properties of the equilibrium point of the system under the condition of nonlinear cost and makes the numerical analysis; Section 4 is the summary of the whole paper.

2 The Model

We consider a differentiated product market with an inverse demand function. It is assumed that the market promotes differentiated products to give consumers certain preferences and their main interests take advantage of differentiated products. The first enterprise produces quantity q_1 at price p_1, and the second enterprise produces quantity q_2 at price p_2. Here, the competitive profile is $p=(p_1,p_2)$, where each firm wants to maximize profits based on the following:

$$\underset{p_i}{Max}\,\pi_i(p_i,p_{-i}) = q_i p_i - C_i(q_i) \tag{1}$$

where p_{-i} refers to the price of another enterprise different from enterprise i, and $C_i(q_i)$ is the cost function. Using the consumer preferences introduced by Singh and Vives, the following utility function is given:

$$U(q_1,q_2) = a(q_1+q_2) - \frac{1}{2}(q_1^2+q_2^2+2bq_1q_2); q_1,q_2>0 \tag{2}$$

Simple calculations show that this function is strictly concave. Using the constraint conditions $\sum_{i=1}^{2} p_i q_i = 1$ and $q_i = \frac{\partial U}{\partial p_i}, i=1,2$, the price function is obtained as follows:

$$\begin{cases} p_1 = a - q_1 - bq_2 \\ p_2 = a - q_2 - bq_1 \end{cases} \tag{3}$$

Parameter a is a positive constant on which the total output of both firms depends and $q_1+q_2=\frac{2a}{1+b}$, where parameter b measures the degree of horizontal differentiation. If $b=0$, then the market is controlled by two monopolies; when b approaches 1, the difference between the two firms becomes smaller. Assume that this parameter is negative, implying complementarity between enterprises.

p_1, q_2 is expressed as a function of q_1, p_2, and the competitive model of q_1, p_2 is studied as follows:

$$\begin{cases} p_1 = a(1-b) - (1-b^2)q_1 + bp_2 \\ \qquad q_2 = a - p_2 - bq_1 \end{cases} \tag{4}$$

3 Complex Dynamic Analysis under Nonlinear Cost

In the game of this work, we consider a nonlinear cost function:

$$C_i(q_i) = c_i q_i + d_i q_i q_j,\quad i, j = 1, 2, i \neq j \tag{5}$$

Net profit is expressed as follows:

$$\begin{aligned} \pi_1(q_1, p_2) &= p_1 q_1 - C_1 = \left[a(1-b) - (1-b^2) q_1 + b p_2 \right] q_1 - c_1 q_1 - d_1 q_1 (a - p_2 - b q_1) \\ \pi_2(q_1, p_2) &= p_2 q_2 - C_2 = p_2 (a - p_2 - b q_1) - c_2 (a - p_2 - b q_1) - d_2 q_1 (a - p_2 - b q_1) \end{aligned} \tag{6}$$

We assume that the firms maximize the relative profit, which is denoted by

$$\begin{aligned} \Pi_1(q_1, p_2) &= \pi_1(q_1, p_2) - \pi_2(q_1, p_2) \\ \Pi_2(q_1, p_2) &= \pi_2(q_1, p_2) - \pi_1(q_1, p_2) \end{aligned} \tag{7}$$

Substituting (6) into (7), after some modifications, we get

$$\begin{aligned} \Pi_1(q_1, p_2) &= G q_1 - H p_2 + E q_1^2 - p_2^2 + F q_1 p_2 \\ \Pi_2(q_1, p_2) &= -G q_1 + H p_2 - E q_1^2 + p_2^2 - F q_1 p_2 \end{aligned} \tag{8}$$

where,

$E = (b^2 - 1) + b(d_1 - d_2)$, $F = 2b + d_1 - d_2$, $G = a(1-b) - (c_1 + bc_2) + (d_2 - d_1)a$, $H = a + c_2$.

According to the relative profit maximization conditions $\frac{\partial \Pi_1}{\partial q_1} = 0, \frac{\partial \Pi_2}{\partial p_2} = 0$, Substituting Equation (8) into them, we can get:

$$\begin{aligned} \frac{\partial \Pi_1}{\partial q_1} &= G + 2E q_1 + F p_2 = 0 \\ \frac{\partial \Pi_2}{\partial p_2} &= -F q_1 - 2 p_2 + H = 0 \end{aligned} \tag{9}$$

To construct and study the dynamic characteristics of this game, we assume that both firms are behaving rationally. Rational behavior implies that firms adopt short-sighted adjustment mechanisms that require firms to know information about whether their relative profits are increasing or decreasing. Using this mechanism, we build a dynamic system to describe the game process as follows:

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha_1 q_1 \frac{\partial \Pi_1}{\partial q_1} \\ p_2(t+1) &= \mathrm{p}_2(t) + \alpha_2 p_2 \frac{\partial \Pi_2}{\partial p_2} \end{aligned} \tag{10}$$

where, α_1,α_2 are the adjustment speed parameters.

Now, by applying steady-state conditions $q_1(t+1)=q_1(t)$ and $p_2(t+1)=p_2(t)$ to each time step t, it is easy to obtain that the above system (10) has the following four equilibrium points:

$$S_1=(0,0),S_2=(-\frac{G}{2E},0),S_3=(0,\frac{H}{2}),S_4=(\overline{q_1},\overline{p_2})$$

where,

$$\overline{q_1}=\frac{(a-c_2)(d_2-d_1)+2(a-c_1)}{4+(d_1-d_2)^2}$$
$$\overline{p_2}=\frac{a(d_2-d_1)^2+\left[(a-c_1)-b(a-c_2)\right](d_2-d_1)+2(a+c_2)-2b(a-c_1)}{4+(d_1-d_2)^2} \qquad (11)$$

In what follows, we assume the following conditions hold

① $a-c_1>b(a-c_2)$

② $-2b<d_1-d_2<\min\{\frac{1-b^2}{b},\frac{2(a-c_1)}{a-c_2}\}$

③ $a>c_1,a>c_2$

It is easy to see that these constrains lead to

$$G>0,E>0,H>0$$
$$\overline{q_1}=\frac{HF+2G}{4+(d_1-d_2)^2}>0$$
$$\overline{p_2}=\frac{FG+2H}{4+(d_1-d_2)^2}>0$$

The above four equilibrium points depend on parameters a,b,c_1,c_2,d_1,d_2, while the other parameters α_1,α_2 also affect their stability. The stability region of the equilibrium point under this model is determined by proving the following propositions.

Proposition 3.1: Equilibrium point e is locally stable if:

$$0<(AB+\mu)\alpha_1\alpha_2<A\alpha_1+B\alpha_2<\frac{1}{2}(AB+\mu)\alpha_1\alpha_2+2 \qquad (12)$$

where,

$$A=-4E\overline{q_1}-F\overline{p_2}-G\,,\quad B=F\overline{q_1}+4\overline{p_2}-H\,,\quad \mu=B^2\overline{q_1}\,\overline{p_2}$$

Proof: To study the stability of the equilibrium point, we continue to consider the aforementioned stability conditions, namely,

$$\begin{cases} 1-T+D>0 \\ 1+T+D>0 \\ 1-D>0 \end{cases} \qquad (13)$$

The Jacobian matrix of system (10) at the equilibrium point is

$$J=\begin{bmatrix} 1+(4E\overline{q_1}+F\overline{p_2}+G)\alpha_1 & F\overline{q_1}\alpha_1 \\ -F\overline{p_2}\alpha_2 & 1+(-F\overline{q_1}-4\overline{p_2}+H)\alpha_2 \end{bmatrix} \qquad (14)$$

the trace and determinant of the matrix become

$$\begin{aligned} T&=2-A\alpha_1-B\alpha_2 \\ D&=(1-A\alpha_1)(1-B\alpha_2)+\mu\alpha_1\alpha_2 \end{aligned} \qquad (15)$$

Therefore, the above stability conditions can be transformed into

$$\begin{aligned} &1-T+D>0\rightarrow(AB+\mu)\alpha_1\alpha_2>0 \\ &1+T+D>0\rightarrow(AB+\mu)\alpha_1\alpha_2-2A\alpha_1-2B\alpha_2+4>0 \\ &1-D>0\rightarrow A\alpha_1+B\alpha_2-(AB+\mu)\alpha_1\alpha_2>0 \end{aligned} \qquad (16)$$

If all three conditions are not met, the equilibrium point will become unstable. Therefore, if any of these conditions is violated, the equilibrium point will lose its stability due to doubling bifurcation or Neimark-Sacker bifurcation. Through simple calculations, these conditions can be transformed into:

$$0<(AB+\mu)\alpha_1\alpha_2<A\alpha_1+B\alpha_2<\frac{1}{2}(AB+\mu)\alpha_1\alpha_2+2 \qquad (17)$$

To sum up, the proposition is proved.

As mentioned above, the three conditions of (13) define a region in the adjustment velocity plane α_1,α_2 , whose shape is the blue area as shown in Fig. 1. The second inequality is a hyperbola. Violation of this inequality will cause the equilibrium to lose stability due to the period doubling bifurcation. It can be observed from the information in Fig. 1 that the local stability of the equilibrium point can be guaranteed in the blue region of the α_1,α_2-plane. Of course, this stable region is also affected by parameters a and b: when b is close to 1 and a is close to 11, the blue region shrinks. When parameters α_1,α_2 are taken from this blue region, the equilibrium point will lose stability due to the period doubling bifurcation.

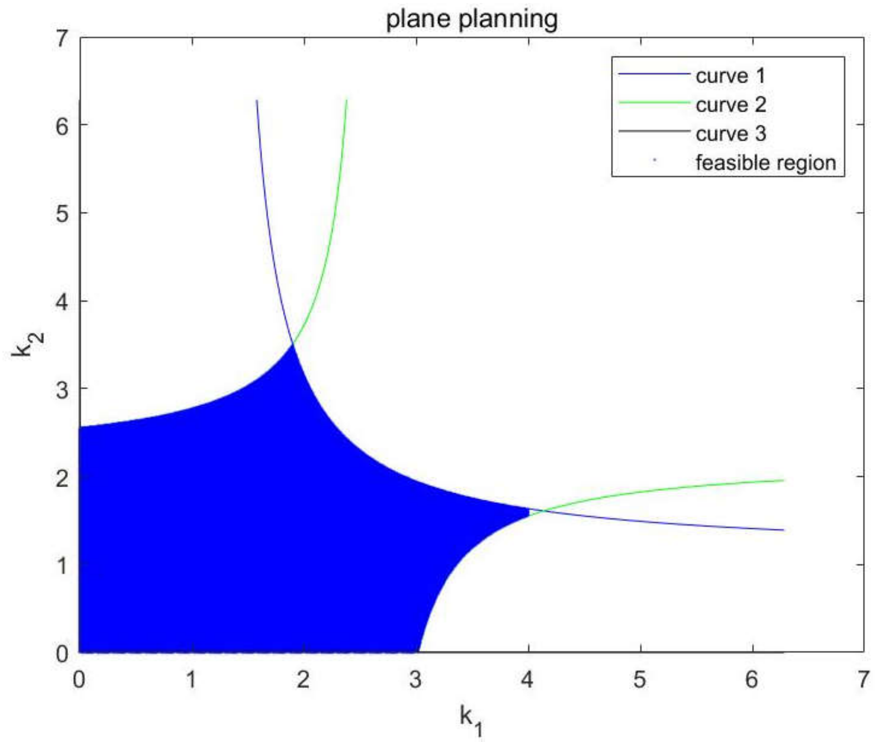

Fig. 1. Equilibrium stability region (The parameter values are $a=0.8, b=0.2, c_1=0.11, c_2=0.12$)

By fixing parameters $b=1, c_1=2, c_2=2, d_1=1, d_2=3, \alpha_1=0.1, \alpha_2=0.15$, the bifurcation diagram of q_1, p_2 with parameter a is obtained, as shown in Fig. 2a, indicating that the occurrence of period-doubling bifurcation leads to the instability of the equilibrium point of the system. When $a \in (0,11)$, the equilibrium point is roughly in a stable state,and when $b>1$, the equilibrium point gradually loses its stability and enters the chaotic region.

Through fixing parameters $a=8, c_1=2, c_2=2, d_1=1, d_2=0.5, \alpha_1=0.1, \alpha_2=0.1$, the bifurcation diagram of q_1, p_2 with parameter b is obtained, as shown in Fig. 2b, when $b \in (0,1)$, the equilibrium point is roughly in a stable state. When $b>1$, the equilibrium point gradually loses its stability and enters the chaotic region.

By fixing parameters $a=10, b=1, c_2=2, d_1=1.5, d_2=3, \alpha_1=0.1, \alpha_2=0.1$, the bifurcation diagram of q_1 with parameter c_1 is obtained as shown in Fig. 2c. When $c_1 \in (0,35)$, the equilibrium point is roughly in a stable state, and as the parameter value gradually increases, when $c_1>35$, the period doubling bifurcation begins to appear, and then the system behavior gradually becomes chaotic.

By fixing parameters $a=10, b=1, c_2=2, d_1=2, d_2=4, \alpha_1=0.1, \alpha_2=0.1$, the bifurcation diagram of p_2 with parameter c_2 is obtained, as shown in Fig. 2d. When $c_2 \in (0,9)$, the equilibrium point is roughly in a stable state, but with the gradual increase of the parameter value, the period doubling bifurcation begins eventually leading to a chaotic construction.

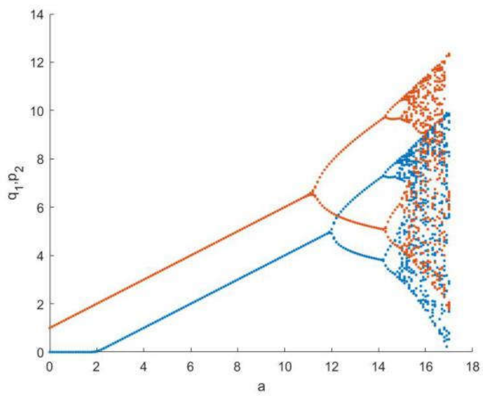

Fig. 2a. q_1, p_2 **image changing with parameters** q_1, p_2

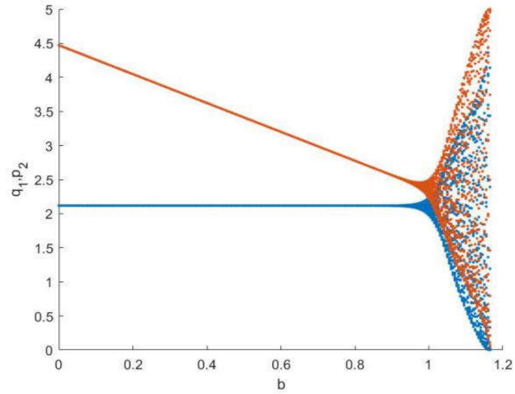

Fig. 2b. q_1, p_2 **image changing with parameters** q_1, p_2

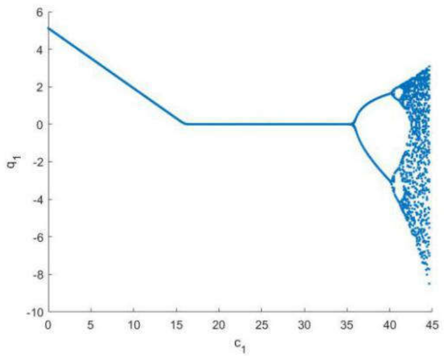

Fig. 2c. q_1 **image changing with parameter** c_1

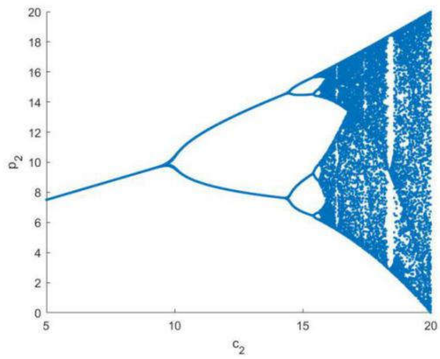

Fig. 2d. p_2 **image changing with parameter** c_2

By fixing parameters $a=10, b=1, c_1=2, c_2=2, d_2=1, \alpha_1=0.1, \alpha_2=0.1$, the bifurcation diagram of q_1, p_2 with respect to parameter d_1 is obtained, as shown in Fig. 2e. When $d_1 \in (0,1)$, the equilibrium point is in a stable state. As the parameter value increases, when the parameter $d_1 > 1$, the period doubling bifurcation appears, which eventually leads to a chaotic construction.

By fixing parameters $a=10, b=1, c_1=2, c_2=2, d_1=1, \alpha_1=0.1, \alpha_2=0.1$, the bifurcation diagram of q_1, p_2 with parameter d_2 is obtained, as shown in Fig. 2f. When the value of parameter d_2 approaches 0, the system behavior is relatively stable; when $b \in (0,1)$, it enters the chaotic region. As the parameter value increases, the system behavior becomes stable when the parameter $b > 1$.

Bifurcation diagram 3g shows the influence of the adjustment speed parameter α_1 of output q_1 on the equilibrium point. When other parameters are $a=10, b=1, c_1=2, c_2=2, d_1=0.1, d_2=2$， $\alpha_2=0.1$, respectively, it can be observed from Fig. 2e that, when $\alpha_1 \in (0, 0.1)$, the equilibrium point is roughly in a stable state, and when $\alpha_1 > 0.1$, the behaviour of the equilibrium point begins to become complex and chaotic.

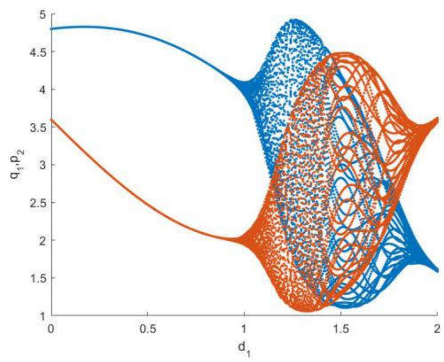

Fig. 2e. q_1, p_2 **image changing with parameter** d_1

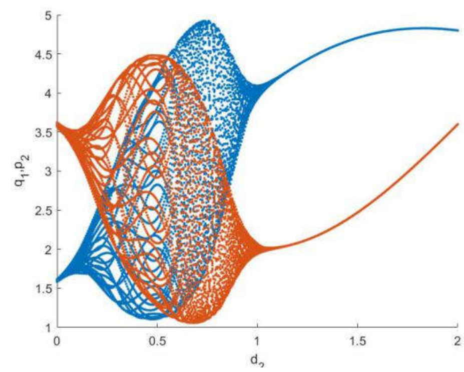

Fig. 2f. q_1, p_2 **image changing with parameter** d_2

Bifurcation diagram 3h shows the influence of the adjustment speed parameter α_2 of price p_2 on the equilibrium point. When other parameters are $a = 5, b = 1, c_1 = 2, c_2 = 1, d_1 = 1, d_2 = 2.5$, $\alpha_2 = 0.2$, respectively, it can be observed from Fig. 2f that, when $\alpha_2 \in (0, 0.4)$, the equilibrium point is roughly in a stable state. As the value of α_2 increases, the period doubling bifurcation begins to appear, and then the system behavior gradually becomes chaotic.

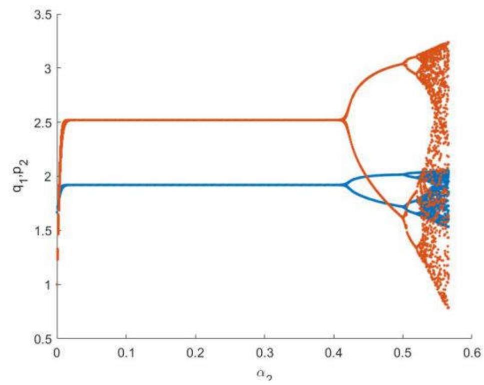

Fig. 2g. q_1, p_2 **image changing with parameter** α_1

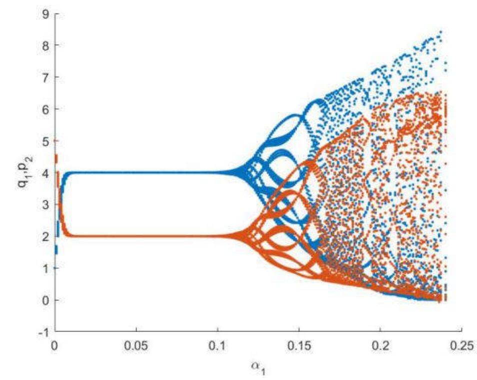

Fig. 2h. q_1, p_2 **image changing with parameter** α_2

4 CONCLUSION

In this paper, we study the complex dynamic behavior of a kind of mixed duopoly game about price and quantity competition. It is assumed that each firm maximizes its expected relative profit under bounded conditions in each period, and the rational expectation of a discrete dynamic system is obtained. This paper studies the construction of a hybrid oligopoly game model whose cost function is a nonlinear function. By analyzing the existence and local stability of the equilibrium point of the dynamic system, single parameter bifurcation diagrams are used to describe the dynamic phenomenon of the system under the changes of product difference, price or quantity adjustment speed. The results show that even a small change in the speed of price and quantity adjustment will have a significant impact on the stability of the system. The reduction of product differentiation will encourage enterprises to increase profits by raising prices, and the change of cost will also have a complex impact on the choice of enterprises. The research on the mixed duopoly game model provides important theoretical support for the enterprises' competitive strategy under the demand of its relative profit maximization.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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