Asian Journal of Probability and Statistics

11(2): 30-46, 2021; Article no.AJPAS.66377

ISSN: 2582-0230

A New Extended Generalized Inverse Exponential Distribution: Properties and Applications

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/AJPAS/2021/v11i230264
Editor(s):
(1) Dr. Seemon Thomas, Mahatma Gandhi University, India.
Reviewers:
(1) Hazem Al-Mofleh, Tafila Technical University, Jordan.
(2) Melanie G. Gurat, Saint Mary's University, Philippines.
(3) Chipepa Fastel, Botswana International University of Science and Technology, Botswana.
Complete Peer review History: http://www.sdiarticle4.com/review-history/66377

Original Research Article

Received: 03 January 2021
Accepted: 09 March 2021
Published: 19 March 2021

Abstract

The quest by researchers in the area of distribution theory in proposing new models with greater flexibility has filled literature. On this note, we proposed a new distribution called the new extended generalized inverse exponential distribution with five positive parameters, which extends and generalizes the extended generalized inverse exponential distribution. We derive some mathematical properties of the proposed model including explicit expressions for the quantile function, moments, generating function, survival, hazard rate, reversed hazard rate, cumulative hazard rate function and odds functions. The method of maximum likelihood is used to estimate the parameters of the distribution. We illustrate its potentiality with applications to three real life data sets which show that the new extended generalized inverse exponential model provides greater flexibility and better fit than other competing models considered.

Keywords: Bio-medical analysis; carbon fibers; generalized inverse exponential; survival times; vinyl chloride.

1 Introduction

Researchers are in the quest of developing and proposing new models in the area of distribution theory by generalizing the existing ones. The generalization is done by adding more parameters to improve the

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flexibility. The literature is filled with such distributions that are very worthwhile in predicting and modeling real life scenario. A number of classical distributions have been used comprehensively over the past decades for modeling data in several applied areas including bio-medical analysis, reliability engineering, economics, forecasting, astronomy, demography and insurance.

In generating new distributions, a number of methods have been proposed. A popular and most used method is the use of family of distributions because it can be used to model a wide variety of random phenomena. Usually the standard distributions will be mathematically simpler, and often other members of the family can be constructed from the standard distributions by simple transformations on the underlying standard random variable. Some recent families of distributions are: Kumaraswamy odd Burr G family of distributions by Nasir et al. [1], the Marshal-Olkin Odd Lindley-G family of distributions by Jamal et al. [2], The Exponentiated Kumaraswamy-G family of distributions by Silva et al. [3], the Topp Leone exponentiated G family of distributions by Ibrahim et al. [4], The Topp Leone Kumaraswamy-G family of distributions by Ibrahim et al. [5], Odd Chen-G family of distributions by Anzagra et al. [6], Modi family of continuous probability distributions by Modi et al. [7].

Despite the usage of exponential distribution in Poisson processes, reliability engineering and its attractive properties, the fact that the exponential distribution has a constant failure rate is a disadvantage because for that singular reason, the distribution becomes unsuitable for modeling real life situations with bathtub and inverted bathtub failure rates [8]. This is actually a serious short-coming of the exponential distribution. Also, the memorylessness is rarely obtainable in real life phenomena. To overcome the limitations of exponential distribution, Keller and Kamath [9] came up with a modified version of the exponential distribution, this modification resulted into the inverse exponential distribution and it has also been studied in details by Lin et al. [10].

Gupta and Kundu [11], generalized the exponential distribution by appending the shape parameter, and named the distribution as the generalized exponential distribution. Generalized inverted exponential distribution was first introduced by Abouammoh and Alshingiti [12]. This distribution originated from the exponentiated Frechet distribution [13].

The generalized inverse exponential distribution provides many practical applications, including, in horse racing, queue theory, modeling wind speeds. Ibrahim et al. [14] extended the generalize inverse exponential distribution by adding one shape parameter to the distribution to make it more flexible in modeling real life data. Oguntunde and Adejumo [15] have explored the statistical properties of the generalized inverted generalized exponential distribution and its parameters were estimated at both censored and uncensored cases using the method of maximum likelihood estimation (MLE). Dey et al. [16] presents some estimation and prediction of unknown parameters based on progressively censored generalized Inverted Exponential data.

The probability density and cumulative density function of generalized Inverted exponential distribution with shape parameter β and scale parameter λ, are given respectively as

$$H(x;\beta,\lambda) = 1 - \left(1 - e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda} \tag{1}$$

and

$$h(x;\beta,\lambda) = \frac{\beta\lambda}{x^2} e^{-\left(\frac{\beta}{x}\right)} \left(1 - e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda-1}, \; x \geq 0. \tag{2}$$

This paper aims to introduce a new extended version of the generalized inverse exponential distribution called new extended generalized inverse exponential distribution using the Topp Leone Kumaraswamy-G (TLK-G) family of distribution. The motivation in proposing this new model is to handle data sets that exhibit skewness, (left-skewed and right-skewed), symmetric or reversed-J shape and to provide consistently better fits than other competing distributions. The rest of the paper is outline as follows. In section 2, the new extended generalized inverse exponential distribution is defined. Linear representation of the new model is presented in section 3. Section 4 provides the statistical properties of the new model. In section 5, the distribution of order statistics is presented. The maximum likelihood estimation is discussed in section 6. Section 7 presented the application of the new model to real data sets. Finally, concluding remark is given in section 8.

2. The New Extended Generalized Inverse Exponential (NEGIEx) Distribution

For an arbitrary baseline cumulative distribution function (cdf) $H(x,\varphi)$, the TLK-G family with three extra positive shape parameters α, θ and σ has cdf and probability density function (pdf) for (x > 0) given by

$$F(x;\alpha,\theta,\sigma,\varphi) = \{1-[1-H(x,\varphi)^{\alpha}]^{2\sigma}\}^{\theta} \tag{3}$$

and

$$f(x;\alpha,\theta,\sigma,\varphi) = 2\alpha\theta\sigma h(x;\ \varphi)H(x;\ \varphi)^{\alpha-1}[1-H(x;\ \varphi)^{\alpha}]^{2\sigma-1}\{1-[1-H(x;\ \varphi)^{\alpha}]^{2\sigma}\}^{\theta-1} \tag{4}$$

$x > 0$ and $\alpha, \theta, \varphi, \sigma > 0$ respectively.

Where $h(x;\varphi) = \frac{dH(x;\varphi)}{dx}$ is the baseline pdf, α, θ and σ are positive shape parameters.

The cdf of the new model is derived by substituting (1) into (3) as

$$F(x;\alpha,\theta,\sigma,\beta,\lambda) = \left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta}, \tag{5}$$

the pdf corresponding to (5) is given as

$$f(x;\alpha,\theta,\sigma,\beta,\lambda) = 2\alpha\theta\sigma\frac{\beta\lambda}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda-1}\left[1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right]^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma-1}$$
$$\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta-1}, \tag{6}$$

where $x \geq 0, \beta > 0$ is the scale parameter and $\alpha, \lambda, \theta, \sigma > 0$ are the shape parameters respectively.

3 Linear Representation of the New Model

Using the series expansion

$$(1-y)^{b-1} = \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\theta)}{i!\Gamma(\theta-i)}y^i \tag{7}$$

Using the last term in (6) in relation to the expansion in (7), we have

$$\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta-1} = \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\theta)}{i!\,\Gamma(\theta-i)}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma i}$$

$$\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma(i+1)-1}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(2\sigma(i+1))}{j!\,\Gamma(2\sigma(i+1)-j)}\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha j}$$

$$\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha(j+1)-1}=\sum_{i=0}^{\infty}\frac{(-1)^{k}\Gamma(\alpha(j+1))}{k!\,\Gamma(\alpha(j+1)-k)}\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda k}$$

$$\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda(k+1)-1}=\sum_{i=0}^{\infty}\frac{(-1)^{l}\Gamma(\lambda(k+1))}{l!\,\Gamma(\lambda(k+1)-l)}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{l+1}$$

substituting back into (6), we have

$$f(x)=2\alpha\theta\sigma\frac{\beta\lambda}{x^{2}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+k+l}\Gamma(\theta)\Gamma(2\sigma(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))}{i!j!k!l!\Gamma(\theta-i)\Gamma(2\sigma(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{l+1} \quad (8)$$

Equation (8) is the expansion for the pdf in (6).

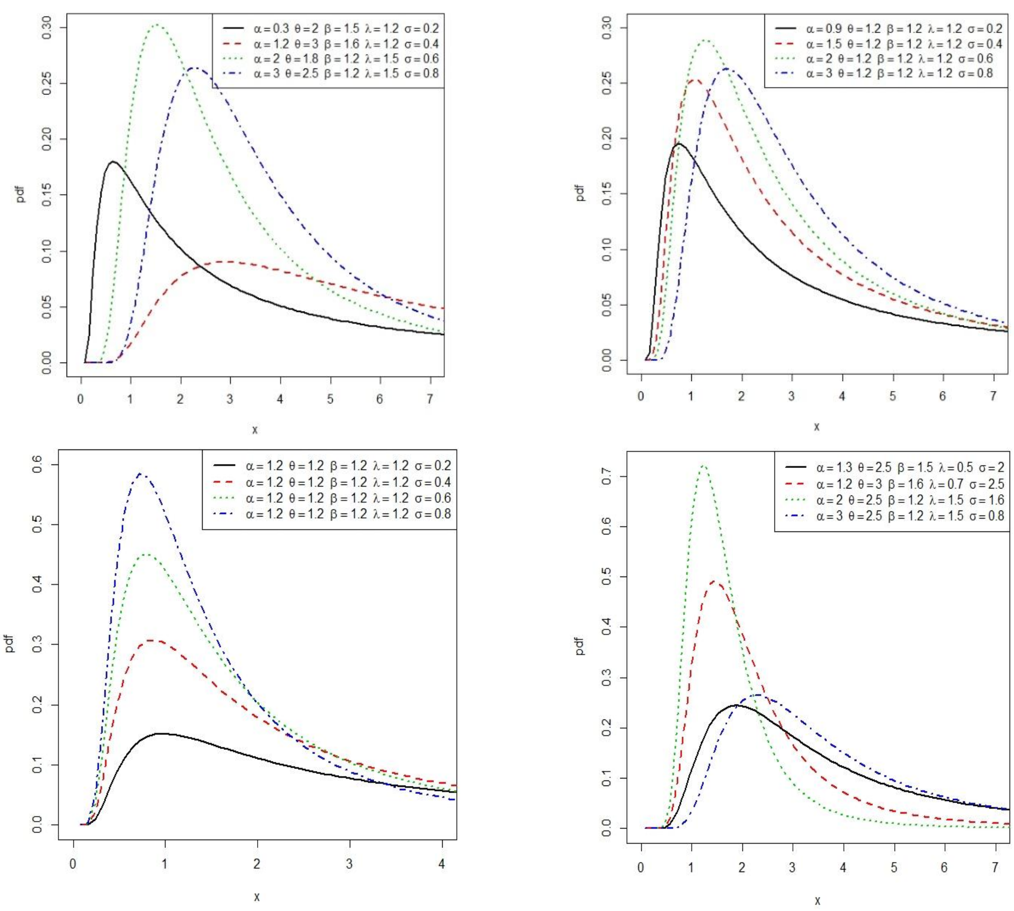

Fig. 1. Plots of pdf of New Extended Generalized Inverse Exponential distribution with different parameter values

4 Statistical Properties

In this section, some of the statistical properties of the new extended generalized inverse exponential distribution will be obtained as follows:

4.1 Moments

Moments function is used to study many important properties of distribution such as dispersion, tendency, skewness and kurtosis. The r^{th} moments of the new extended generalized inverse exponential distribution is obtained as follow:

$$E(X^r) = \int_0^\infty x^r f(x)dx \tag{9}$$

Using (6), we have,

$$E(X^r) = \int_0^\infty x^r\, 2\alpha\theta\sigma \frac{\beta\lambda}{x^2} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{i+j+k+l}\Gamma(\theta)\Gamma(2\sigma(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))}{i!j!k!l!\Gamma(\theta-i)\Gamma(2\sigma(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)} \left[e^{-\left(\frac{\beta}{x}\right)}\right]^{l+1} dx \tag{10}$$

$$E(X^r) = 2\alpha\theta\beta\lambda\sigma \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{i+j+k+l}\Gamma(\theta)\Gamma(2\sigma(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))}{i!j!k!l!\Gamma(\theta-i)\Gamma(2\sigma(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)} \int_0^\infty x^{r-2} \left[e^{-\left(\frac{\beta}{x}\right)}\right]^{l+1} \tag{11}$$

$$E(X^r) = 2\alpha\theta\beta^r\lambda\sigma \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{i+j+k+l}\Gamma(\theta)\Gamma(2\sigma(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))(l+1)^{r-1}\Gamma(1-\mathrm{r})}{i!j!k!l!\Gamma(\theta-i)\Gamma(2\sigma(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)} \tag{12}$$

To obtain the mean, we set $r = 1$ in (12)

4.2 Moment generating function

The moment generating function (mgf) of X can be obtained using the equation

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx \tag{13}$$

$$e^{tx} = \sum_{m=0}^\infty \frac{t^m x^m}{m!} \tag{14}$$

$$M_x(t) = 2\alpha\theta\beta^m\lambda\sigma \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \sum_{m=0}^\infty \frac{(-1)^{i+j+k+l}\Gamma(\theta)\Gamma(2\sigma(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))t^m(l+1)^{m-1}\Gamma(1-\mathrm{m})}{i!j!k!l!m!\Gamma(\theta-i)\Gamma(2\sigma(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)} \tag{15}$$

4.3 Quantile function

The Quantile function is given by;

$$Q(u) = F^{-1}(u) \tag{16}$$

Therefore, the corresponding quantile function for the extended generalized inverse exponenetial model is given by;

$$x = Q(u) = \beta\left[-log\left\{1-\left[1-\left(1-\left(1-u^{\frac{1}{\theta}}\right)^{\frac{1}{2\sigma}}\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{\lambda}}\right\}\right]^{-1} \tag{17}$$

where U has the uniform $U(0,1)$ distribution.

4.4 Median

The median of the extended generalized inverse exponential distribution is obtained by setting $U = 0.5$ in (17) to obtain,

$$x_m = Q(0.5) = \beta\left[-log\left\{1-\left[1-\left(1-\left(1-0.5^{\frac{1}{\theta}}\right)^{\frac{1}{2\sigma}}\right)^{\frac{1}{\alpha}}\right]^{\frac{1}{\lambda}}\right\}\right]^{-1} \tag{18}$$

4.5 Survival function

The survival function, which is the probability of an item not failing prior to some time. It can be defined as

$$S(x) = 1 - F(x) \tag{19}$$

The survival function the NEGIEx distribution is given as

$$S(x) = 1-\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta} \tag{20}$$

4.6 Hazard rate function

The hazard rate function is given as

$$\tau(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} \tag{21}$$

$$\tau(x) = \frac{2\alpha\theta\frac{\beta\lambda}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda-1}\left[1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right]^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma-1}\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta-1}}{1-\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta}} \tag{22}$$

4.7 Odds function

The odds function is obtained using the relation

$$Q(x) = \frac{F(x)}{S(x)} \tag{23}$$

$$Q(x) = \frac{\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta}}{1-\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta}} \tag{24}$$

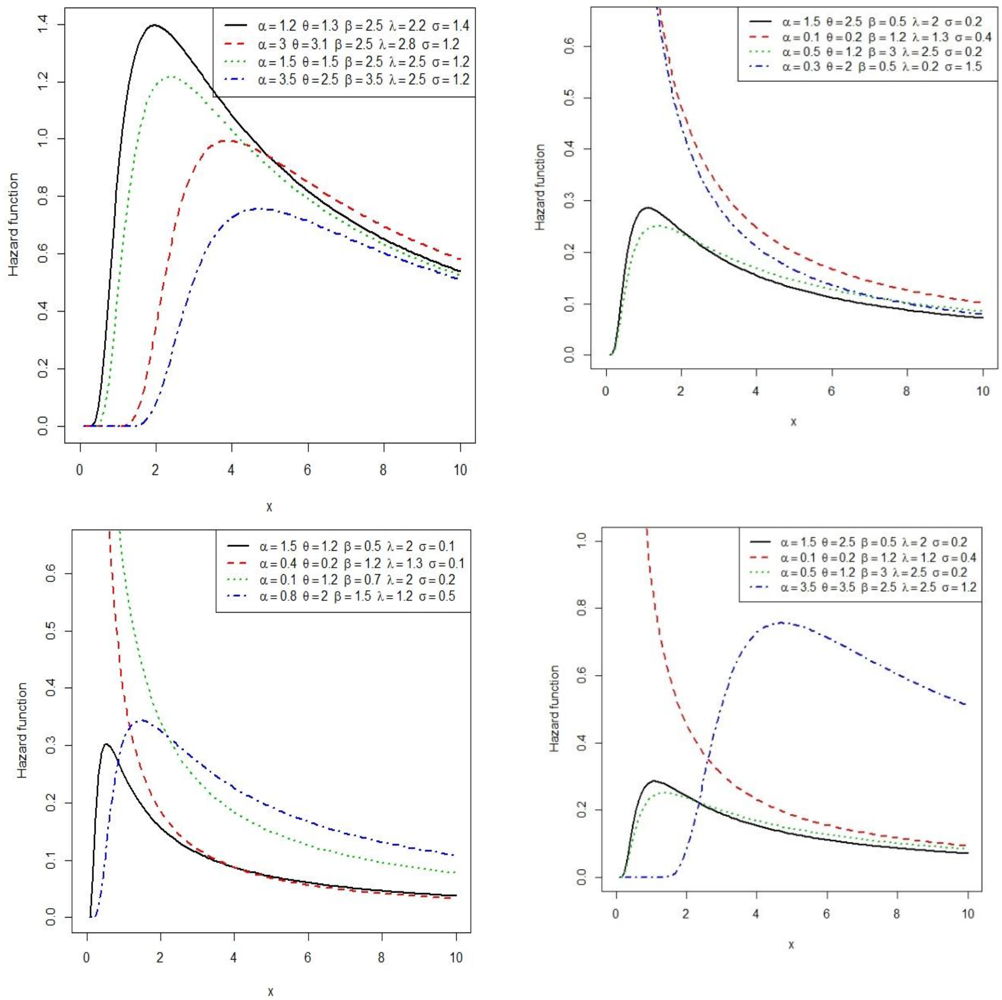

Fig. 2. Plots of hrf of New Extended Generalized Inverse Exponential distribution with different parameter values

From Fig. 2, it can be seen that the new model exhibits different shapes such as increasing, decreasing, J-shape and reversed J-shape.

4.8 Reversed hazard rate function

The reverse hazard rate function of the new extended generalized inverse exponential distribution is given as

$$\emptyset(x) = \frac{f(x)}{F(x)} \tag{25}$$

$$\emptyset(x) = \frac{2\alpha\theta\frac{\beta\lambda}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda-1}\left[1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right]^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma-1}\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta-1}}{\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta}} \tag{26}$$

4.9 Cumulative hazard function

$$C(x) = -\ln\left(S(x)\right) \tag{27}$$

Then, the cumulative hazard function of the NEGIEx distributions is given as

$$C(x) = -\ln\left[1-\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\}^{\theta}\right] \tag{28}$$

5 Distribution of Order Statistics

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample and its ordered values are denoted as $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$. The pdf of order statistics is obtained using the below function

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \tag{29}$$

5.1 Minimum order statistics

The minimum order statistics is obtained by setting $r = 1$ in (29) as

$$f_{1:n}(x) = nf(x)[1-F(x)]^{n-1} \tag{30}$$

Then the minimum order statistics of the new extended generalized inverse exponential distribution is given as

$$f_{1:n}(x) = 2n\alpha\theta\sigma\beta\lambda \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j+k+l+m}\Gamma(n)\Gamma(\theta(i+1))\Gamma(2\sigma(j+1))\Gamma(\alpha(k+1))\Gamma(\lambda(l+1))x^{-2}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{m+1}}{i!j!k!l!m!\Gamma(n-i)\Gamma(\theta(i+1)-j)\Gamma(2\sigma(j+1)-k)\Gamma(\alpha(k+1)-l)\Gamma(\lambda(l+1)-m)} \tag{31}$$

5.2 Maximum order statistics

The maximum order statistics is obtained by setting $r = n$ in (29) as

$$f_{n:n}(x) = nf(x)[F(x)]^{n-1} \tag{32}$$

Then the maximum order statistics of the new extended generalized inverse exponential distribution is given as

$$f_{n:n}(x) = 2n\alpha\theta\sigma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+j+k+l}\Gamma(\theta(n-1)+1)\Gamma(2\sigma(i+1))\Gamma(\alpha(j+1))\Gamma(\lambda(k+1))}{i!j!k!l!\Gamma(\theta(n-1)+-i)\Gamma(2\sigma(i+1)-j)\Gamma(\alpha(j+1)-k)\Gamma(\lambda(k+1)-l)} \frac{\beta\lambda}{x^2}\left[e^{-\left(\frac{\beta}{x}\right)}\right]^{l+1} \tag{33}$$

6 Maximum Likelihood Estimates

Since maximum likelihood estimators give the maximum information about the population parameters, therefore this section presents the maximum likelihood estimates (MLEs) of the parameters that are inherent within the new extended generalized inverse exponential distribution function given by the following: Let $X_1, X_2, X_3, \ldots, X_n$ be random variables of the extended generalized inverse exponential distribution of size n. Then sample likelihood function of extended generalized inverse exponential distribution is obtained as

$$L(x) = (2\alpha\theta\beta\lambda\sigma)^n \prod_{i=1}^{n} \left\{ \frac{1}{x_i^2} e^{-\left(\frac{\beta}{x_i}\right)} \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda-1} \left[1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right]^{\alpha-1} \left[1 - \left(1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma-1} \left\{ 1 - \right.\right.$$

1–1–1–e–βxiλα2σθ–1 (34)

Log-likelihood function $logL(x)$ is

$$logL(x) = nlog2 + nlog\alpha + nlog\theta + +nlog\sigma + nlog\beta + nlog\lambda + \sum_{i=1}^{n} log\left(\frac{1}{x_i^2}\right) - \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right) + (\lambda - 1)\sum_{i=1}^{n} log\left(1 - \right.$$

e–βxi+α–1i=1nlog1–1–e–βxiλ+(2σ–1)i=1nlog1–1–1–e–βxiλα+(θ–1)i=1nlog1–1–1–1–e–βxiλα2σ
(35)

Therefore, The MLE's of parameters $\alpha, \beta, \theta, \sigma, \lambda$ which maximize the above log-likelihood function must satisfy the normal equations. We take the first derivative of the above log-likelihood equation with respect to each parameter and equate to zero respectively.
\

$$\frac{\partial logL(x)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\left(1 - \left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right) + (2\sigma - 1)\sum_{i=1}^{n} \frac{\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha} log\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)}{1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}} + (\theta -$$

$$1)\sum_{i=1}^{n} \frac{2\sigma\left[\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma-1} log\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)}{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}} \quad (36)$$

$$\frac{\partial logL(x)}{\partial \beta} =$$

$$\frac{n}{\beta} - \sum_{i=1}^{n}\left(\frac{1}{x_i}\right) + (\lambda - 1)\sum_{i=1}^{n} \frac{\left(\frac{1}{x_i}\right)e^{-\left(\frac{\beta}{x_i}\right)}}{\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)} + (\alpha - 1)\sum_{i=1}^{n} \frac{\lambda\left(\frac{1}{x_i}\right)e^{-\left(\frac{\beta}{x_i}\right)}\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)}{1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}} + (2\sigma -$$

$$1)\sum_{i=1}^{n} \frac{\alpha\lambda\left(\frac{1}{x_i}\right)e^{-\left(\frac{\beta}{x_i}\right)}\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda-1}\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha-1}}{1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}} + (\theta -$$

$$1)\sum_{i=1}^{n} \frac{2\alpha\lambda\sigma\left(\frac{1}{x_i}\right)e^{-\left(\frac{\beta}{x_i}\right)}\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda-1}\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma-1}}{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}} \quad (37)$$

$$\frac{\partial logL(x)}{\partial \lambda} =$$

$$\frac{n}{\lambda} + \sum_{i=1}^{n} log\left(1 - e^{-\left(\frac{\beta}{x_i}\right)}\right) + (\alpha - 1)\sum_{i=1}^{n} \frac{\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda} log\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)}{1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}} + (2\sigma -$$

$$1)\sum_{i=1}^{n}\frac{\alpha\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda} log\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha-1}}{1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}}+(\theta-$$

$$1)\sum_{i=1}^{n}\frac{2\alpha\sigma\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma-1} log\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)}{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}} \tag{38}$$

$$\frac{\partial logL(x)}{\partial\sigma}=\frac{n}{\sigma}+2\sum_{i=1}^{n} log\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]+(\theta-1)\sum_{i=1}^{n} log\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\} \tag{39}$$

$$\frac{\partial logL(x)}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n} log\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x_i}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2\sigma}\right\} \tag{40}$$

Since the above derived equations (36), (37), (38), (39) and (40) are in the complex form, therefore the exact solution of ML estimator for unknown parameters is not possible. So it is convenient to use nonlinear Newton Raphson algorithm for exact numerically solution to maximize the above likelihood function.

7 Applications

In this section, we applied two data sets to illustrate the usefulness of the proposed model and observe its flexibility over some existing models. The models considered are:

- Extended generalized inverse exponential (EGIEx) distribution

$$f=2\alpha\theta\frac{\beta\lambda}{x^2}e^{-\left(\frac{\beta}{x}\right)}\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda-1}\left[1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right]^{\alpha-1}\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]\left\{1-\left[1-\left(1-\left(1-e^{-\left(\frac{\beta}{x}\right)}\right)^{\lambda}\right)^{\alpha}\right]^{2}\right\}^{\theta-1}$$

- Inverse exponential (IEx) distribution

$$f(x)=\frac{\beta}{x^2}e^{\left(-\frac{\beta}{x}\right)}$$

- Exponentiated generalized inverse exponential (ExGIEx) distribution

$$f(x)=\frac{\beta\alpha\theta}{x^2}e^{\left(-\frac{\beta}{x}\right)}\left[1-e^{\left(-\frac{\beta}{x}\right)}\right]^{\alpha-1}\left[1-\left(1-e^{\left(-\frac{\beta}{x}\right)}\right)^{\alpha}\right]^{\theta-1}$$

- Generalized Inverse exponential (GIEx) distribution

$$f(x) = \frac{\beta\lambda}{x^2} e^{\left(-\frac{\beta}{x}\right)} \left[1 - e^{\left(-\frac{\beta}{x}\right)}\right]^{\lambda-1}$$

The statistics used in comparing the fit of the models is Akaike Information Criteria (AIC). The model with the lowest AIC is considered the best with regard to the data set considered.

7.1 Data set 1

The first data set represents the breaking stress of carbon fibers of 50 mm length (GPa) was reported by Nicholas and Padgett [17]. This data was used by Yousof et al. [18]. The data are:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 1. MLEs and selection criteria for data set 1

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\sigma}$	$-l$	AIC
NEGIEx	0.0039	10.1562	13.6247	79.2456	0.2025	86.0794	182.1587
EGIEx	1.18861	13.0965	0.3690	19.5331	-	93.9751	195.9502
IEx	-	2.2992	-	-	-	136.0285	274.0570
ExGIEx	39.7875	13.8792	0.4454	-	-	94.0114	196.0228
GIEx	-	13.2879	-	7.6019	-	99.6202	203.2403

Table 1 shows the result of the analysis of data set representing the breaking stress of carbon fibers of 50 mm length (GPa). It can be seen from the result in the table that the NEGIEx distribution has the lowest AIC which makes it fits better and appropriate in this data set than the other competing models considered.

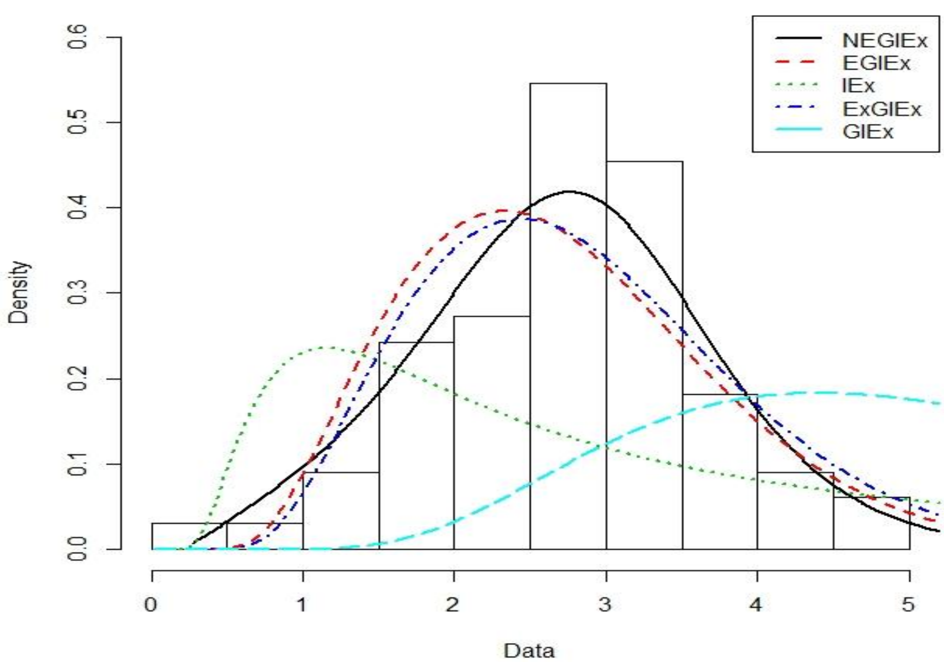

Fig. 3. Histogram and fitted models breaking stress of carbon fibers of 50 mm length (GPa) data

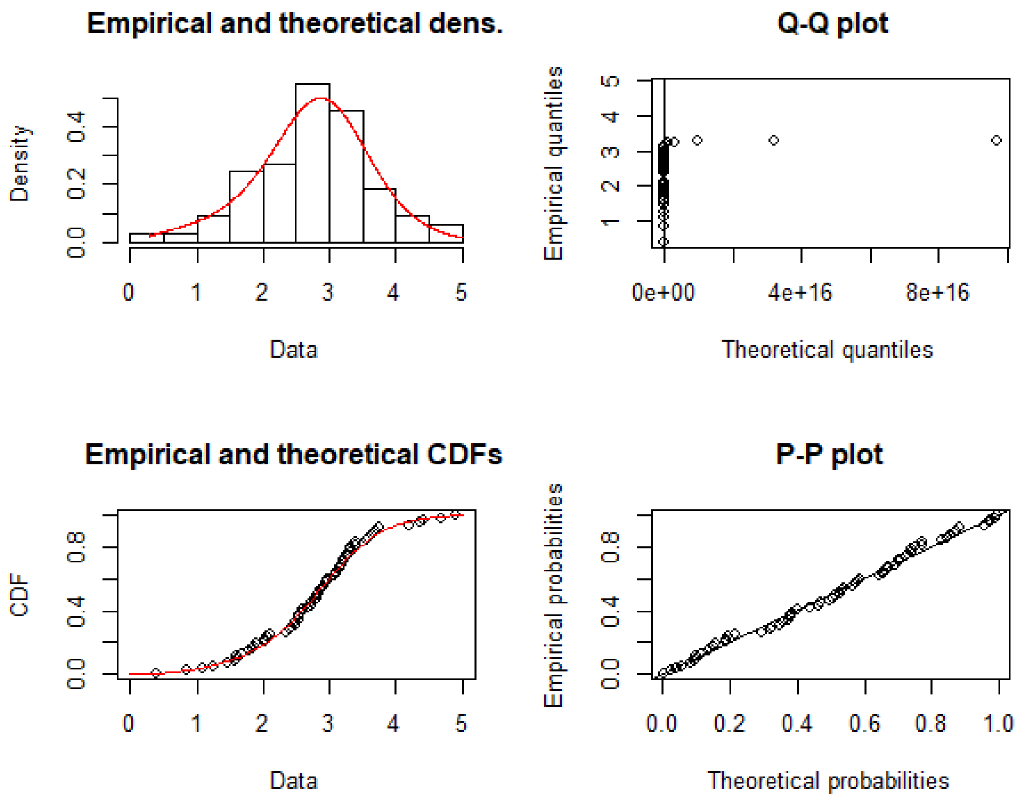

Fig. 4. Plots for the fitted pdf, cdf, Q-Q plot and P-P plot for data set 1

7.2 Data set 2

The second data set was given by Lee [19] and it represents the survival times of one hundred and twenty-one (121) patients with breast cancer obtained from a large hospital in a period from 1929 to 1938. It has also been applied by Ramos et al. [20]. The data set is as follows:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

Table 2 shows the result of the analysis of data set representing the survival times of one hundred and twenty-one (121) patients with breast cancer. It can be seen from the result in the table that the NEGIEx distribution has the lowest AIC which makes it fits better and appropriate in this data set than the other competing models considered.

Table 2. MLEs and selection criteria for data set 2

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\sigma}$	$-l$	AIC
NEGIEx	7.1153	0.0017	7.5251	0.1364	9.9600	609.4956	1228.9910
EGIEx	63.7813	0.0040	2.2264	0.5013	-	622.6595	1253.3190
IEx	-	10.3215	-	-	-	677.2791	1357.5580
ExGIEx	0.2142	0.0021	4.6702	-	-	743.3310	1492.6620
GIEx	-	0.5595	-	6.4034	-	664.1239	1332.2480

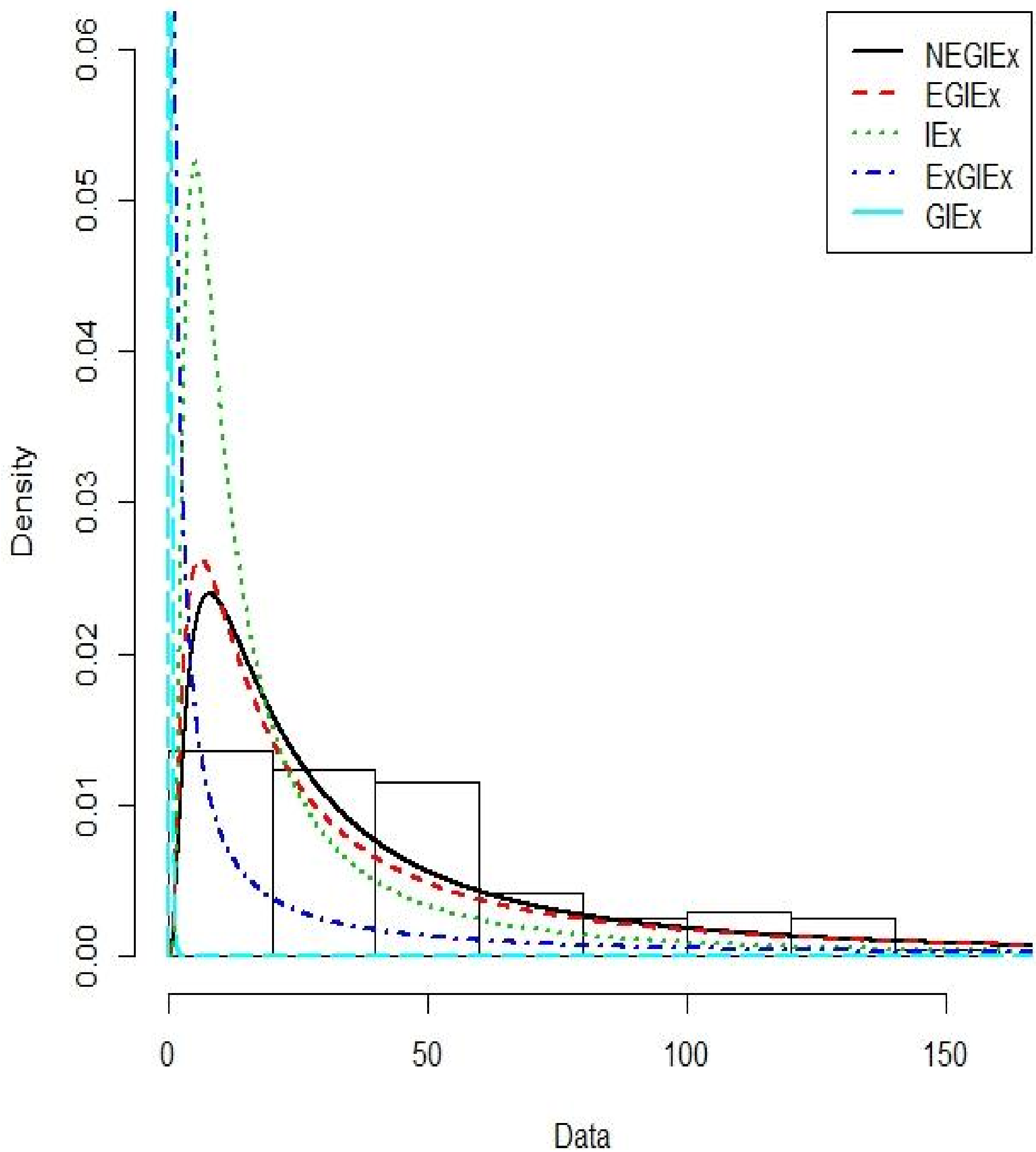

Fig. 5. Histogram and fitted models from survival times of one hundred and twenty-one (121) patients with breast cancer data

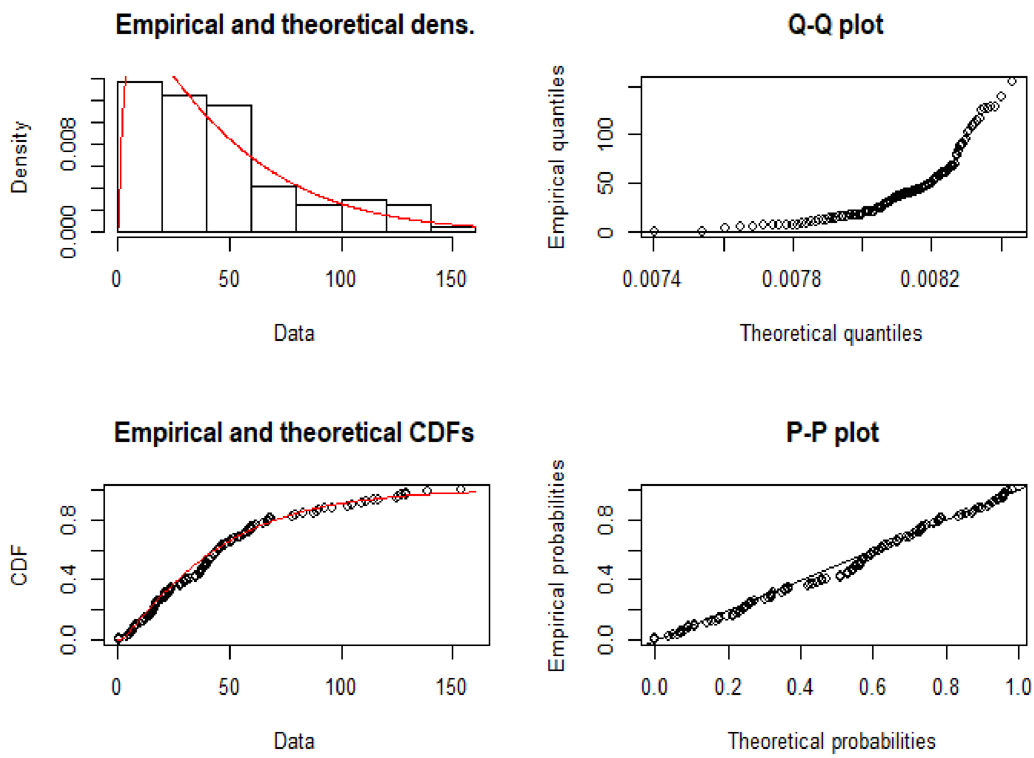

Fig. 6. Plots for the fitted pdf, cdf, Q-Q plot and P-P plot for data set 2

7.3 Data set 3

The third data set represents the vinyl chloride (in mg/l) that was obtained from clean up gradient monitoring wells. It has previously been used by Bhaumik et al. [21]. The data set has 34 observations and are presented below:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2.

Table 3. MLEs and selection criteria for data set 1

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\sigma}$	$-l$	AIC
NEGIEx	16.4598	0.0286	0.3223	0.3290	20.7196	54.5905	119.1810
EGIEx	2.5246	1.2638	0.1792	0.7188	-	58.4206	124.8411
IEx	-	0.5725	-	-	-	59.1930	120.3860
ExGIEx	0.8710	0.0056	59.1930	-	-	58.6960	123.3920
GIEx	-	0.9214	-	0.5415	-	59.1285	122.2569

Table 3 shows the result of the analysis of data set representing the vinyl chloride (in mg/l). It can be seen from the result in the table that the NEGIEx distribution has the lowest AIC which makes it fits better and appropriate in this data set than the other competing models considered.

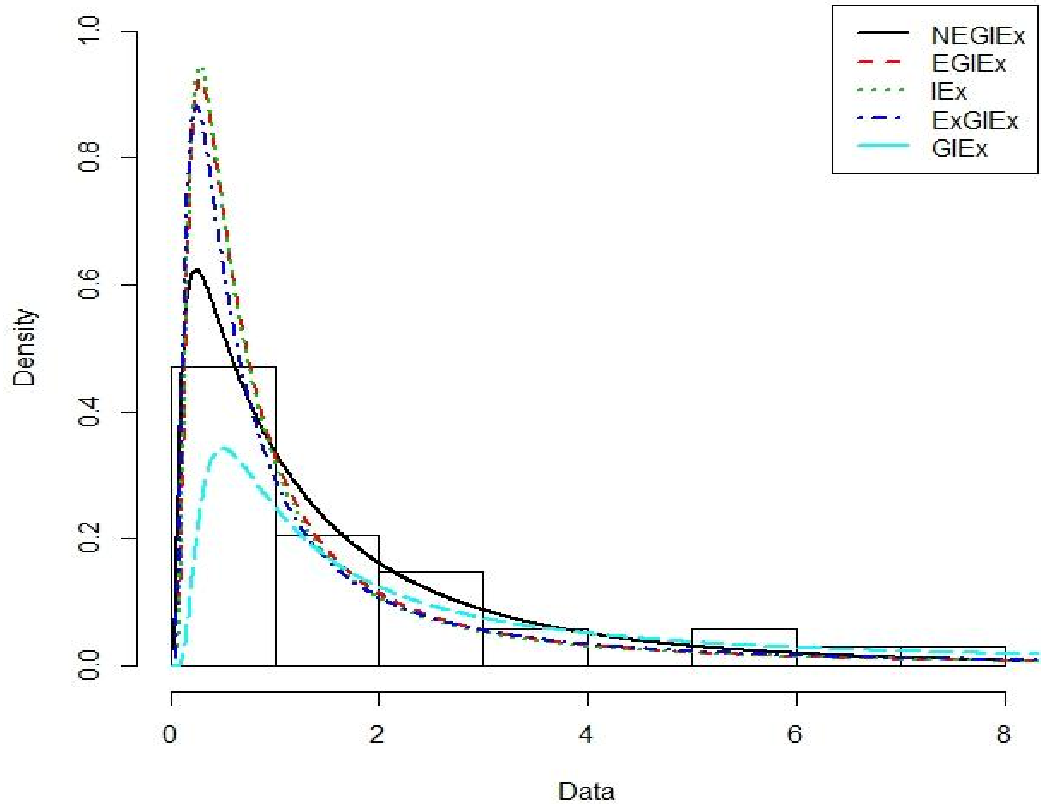

Fig. 7. Histogram and fitted models from vinyl chloride data

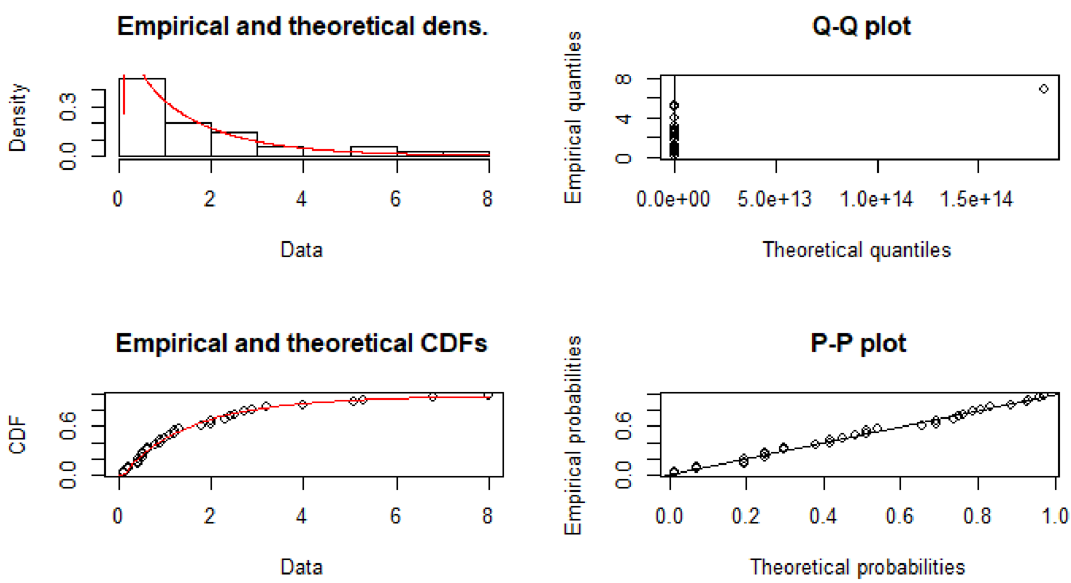

Fig. 8. Plots for the fitted pdf, cdf, Q-Q plot and P-P plot for data set 3

8 Conclusion

In this paper, a five parameter life time model called the new extended generalized inverse exponential distribution which generalizes the extended generalized inverse exponential distribution has been studied. This new model is capable of modeling different kinds of data with different shapes as shown in Fig. 1 and 2. We derived explicit expressions for some of its statistical and mathematical properties including the moments, generating function, quantile function, survival function, hazard rate function, reversed hazard rate function, cumulative hazard rate function and odds function. The minimum and maximum distributions of order statistics of the new model were derived. The model parameters were estimated by using maximum likelihood method based on complete sample. We observed from the analysis that the new extended generalized inverse exponential distribution provides better fits than the competing distributions considered on the three real life data sets based on the value of AIC and also from the histograms and the fitted plots of the pdfs.

Competing Interests

Author has declared that no competing interests exist.

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