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Recurrence Properties of Generalized Hexanacci Sequence

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, we investigate the recurrence properties of the generalized Hexanacci sequence under the mild assumption that the roots of the corresponding characerteristic polynomial are all distinct, and present how the generalized Hexanacci sequence at negative indices can be expressed by the sequence itself at positive indices.

Keywords: Hexanacci numbers; Hexanacci sequence; negative indices; recurrence relations.

2010 Mathematics Subject Classification: 11B37, 11B39, 11B83.

1 INTRODUCTION

We can propose an open problem as follows: Whether and how can the generalized Hexanacci sequence W_n at negative indices be expressed by the sequence itself at positive indices?

We present our main result as follows which completely solves the above problem (under the assumption that the roots of the corresponding characerteristic polynomial are all distinct) for the generalized Hexanacci sequence W_n.

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Theorem 1.1. *For $n \in \mathbb{Z}$, we have*

$W_{-n} = \frac{1}{120}(-y)^{-n}(W_0H_n^5 - 5W_nH_n^4 + 20H_n^3W_{2n} - 10W_0H_n^3H_{2n} + 15W_0H_{2n}^2H_n - 15W_nH_{2n}^2 + 20W_0H_n^2H_{3n} + 30W_nH_n^2H_{2n} - 60H_n^2W_{3n} - 120W_{5n} + 120H_nW_{4n} + 60H_{2n}W_{3n} + 40H_{3n}W_{2n} + 24W_0H_{5n} + 30H_{4n}W_n - 20W_0H_{3n}H_{2n} - 30H_{4n}H_nW_0 - 40H_{3n}H_nW_n - 60H_{2n}H_nW_{2n})$

$= (-1)^{-n-1}y^{-n}(W_{5n} - H_nW_{4n} + \frac{1}{2}(H_n^2 - H_{2n})W_{3n} - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_{2n} + \frac{1}{24}(H_n^4 + 3H_{2n}^2 - 6H_n^2H_{2n} - 6H_{4n} + 8H_{3n}H_n)W_n - \frac{1}{120}(H_n^5 - 10H_n^3H_{2n} + 15H_{2n}^2H_n + 20H_n^2H_{3n} + 24H_{5n} - 30H_{4n}H_n - 20H_{3n}H_{2n})W_0).$

Note that H_n can be written in terms of W_n using Remark 2.1 below.

The generalized (r, s, t, u, v, y) sequence (or the generalized Hexanacci sequence or 6-step Fibonacci sequence) $\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r, s, t, u, v, y)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined by the sixth-order recurrence relation

$$\begin{aligned} W_n &= rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}, \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2,\ W_3 = c_3, W_4 = c_4, W_5 = c_5,\ n \geq 6 \end{aligned} \tag{1.1}$$

where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and r, s, t, u, v, y are real numbers with $y \neq 0$. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5} + \frac{1}{y}W_{-n+6}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integers n. Hexanacci sequence has been studied by many authors, see for example [1,2,3] and references therein.

In the following Table 1 we present some special cases of generalized Hexanacci sequence.

Table 1. A few special case of generalized Hexanacci sequences

No	Sequences (Numbers)	Notation
1	Generalized Hexanacci	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 1, 1, 1, 1, 1, 1)\}$
2	Generalized Sixth Order Pell	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 2, 1, 1, 1, 1, 1)\}$
3	Generalized Sixth Order Jacobsthal	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 1, 1, 1, 1, 1, 2)\}$
4	Generalized 6-primes	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 2, 3, 5, 7, 11, 13)\}$

In literature, for example, the following names and notations (see Table 2) are used for the special case of r, s, t, u, v, y and initial values.

Table 2. A few special case of generalized Pentanacci sequences

Sequences (Numbers)	Notation	OEIS [4]	Ref
Hexanacci	$\{H_n\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 1)\}$	A001592	[5]
Hexanacci-Lucas	$\{E_n\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 1)\}$	A074584	[5]
sixth order Pell	$\{P_n^{(6)}\} = \{W_n(0, 1, 2, 5, 13, 34; 2, 1, 1, 1, 1, 1)\}$		[6]
sixth order Pell-Lucas	$\{Q_n^{(6)}\} = \{W_n(6, 2, 6, 17, 46, 122; 2, 1, 1, 1, 1, 1)\}$		[6]
modified sixth order Pell	$\{E_n^{(6)}\} = \{W_n(0, 1, 1, 3, 8, 21; 2, 1, 1, 1, 1, 1)\}$		[6]
sixth order Jacobsthal	$\{J_n^{(6)}\} = \{W_n(0, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 2)\}$		[7,8]
sixth order Jacobsthal-Lucas	$\{j_n^{(6)}\} = \{W_n(2, 1, 5, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$		[7,8]
modified sixth order Jacobsthal	$\{K_n^{(6)}\} = \{W_n(3, 1, 3, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$		[7]
sixth-order Jacobsthal Perrin	$\{Q_n^{(6)}\} = \{W_n(3, 0, 2, 8, 16, 32; 1, 1, 1, 1, 1, 2)\}$		[7]
adjusted sixth-order Jacobsthal	$\{S_n^{(6)}\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 2)\}$		[7]
modified sixth-order Jacobsthal-Lucas	$\{R_n^{(6)}\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 2)\}$		[7]
6-primes	$\{G_n\} = \{W_n(0, 0, 0, 0, 1, 2; 2, 3, 5, 7, 11, 13)\}$		[9]
Lucas 6-primes	$\{H_n\} = \{W_n(6, 2, 10, 41, 150, 542; 2, 3, 5, 7, 11, 13)\}$		[9]
modified 6-primes	$\{E_n\} = \{W_n(0, 0, 0, 0, 1, 1; 2, 3, 5, 7, 11, 13)\}$		[9]

Here, OEIS stands for On-line Encyclopedia of Integer Sequences. For easy writing, from now on, we drop the superscripts from the sequences, for example we write J_n for $J_n^{(6)}$.

It is well known that the generalized (r,s,t,u,v,y) numbers (the generalized Pentanacci numbers) can be expressed, for all integers n, using Binet's formula

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n + A_6\mu^n$$

where

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)},\\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)},\\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)},\\
A_4 &= \frac{p_4}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)},\\
A_5 &= \frac{p_5}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)},\\
A_6 &= \frac{p_6}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}.
\end{aligned}$$

and

$$\begin{aligned}
p_1 &= W_5-(\beta+\gamma+\delta+\lambda+\mu)W_4+(\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\beta\delta+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3\\
&\quad -(\beta\lambda\gamma+\beta\lambda\mu+\beta\lambda\delta+\beta\gamma\mu+\lambda\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)W_2\\
&\quad +(\beta\lambda\gamma\mu+\beta\lambda\gamma\delta+\beta\lambda\mu\delta+\beta\gamma\mu\delta+\lambda\gamma\mu\delta)W_1-\beta\lambda\gamma\mu\delta W_0,\\
p_2 &= W_5-(\alpha+\gamma+\delta+\lambda+\mu)W_4+(\alpha\lambda+\alpha\gamma+\alpha\mu+\alpha\delta+\lambda\gamma+\lambda\mu+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3\\
&\quad -(\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\lambda\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\lambda\gamma\mu+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)W_2\\
&\quad +(\alpha\lambda\gamma\mu+\alpha\lambda\gamma\delta+\alpha\lambda\mu\delta+\alpha\gamma\mu\delta+\lambda\gamma\mu\delta)W_1-\alpha\lambda\gamma\mu\delta W_0,\\
p_3 &= W_5-(\alpha+\beta+\delta+\lambda+\mu)W_4+(\alpha\beta+\alpha\lambda+\alpha\mu+\beta\lambda+\alpha\delta+\beta\mu+\lambda\mu+\beta\delta+\lambda\delta+\mu\delta)W_3\\
&\quad -(\alpha\beta\lambda+\alpha\beta\mu+\alpha\lambda\mu+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\mu+\alpha\mu\delta+\beta\lambda\delta+\beta\mu\delta+\lambda\mu\delta)W_2\\
&\quad +(\alpha\beta\lambda\mu+\alpha\beta\lambda\delta+\alpha\beta\mu\delta+\alpha\lambda\mu\delta+\beta\lambda\mu\delta)W_1-\alpha\beta\lambda\mu\delta W_0,\\
p_4 &= W_5-(\alpha+\beta+\gamma+\lambda+\mu)W_4+(\alpha\beta+\alpha\lambda+\alpha\gamma+\alpha\mu+\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\gamma\mu)W_3\\
&\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\beta\mu+\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\gamma\mu+\beta\lambda\gamma+\beta\lambda\mu+\beta\gamma\mu+\lambda\gamma\mu)W_2\\
&\quad +(\alpha\beta\lambda\gamma+\alpha\beta\lambda\mu+\alpha\beta\gamma\mu+\alpha\lambda\gamma\mu+\beta\lambda\gamma\mu)W_1-\alpha\beta\lambda\gamma\mu W_0,\\
p_5 &= W_5-(\alpha+\beta+\gamma+\delta+\mu)W_4+(\alpha\beta+\alpha\gamma+\alpha\mu+\alpha\delta+\beta\gamma+\beta\mu+\beta\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3\\
&\quad -(\alpha\beta\gamma+\alpha\beta\mu+\alpha\beta\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\beta\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\gamma\mu\delta)W_2\\
&\quad +(\alpha\beta\gamma\mu+\alpha\beta\gamma\delta+\alpha\beta\mu\delta+\alpha\gamma\mu\delta+\beta\gamma\mu\delta)W_1-\alpha\beta\gamma\mu\delta W_0,\\
p_6 &= W_5-(\alpha+\beta+\gamma+\delta+\lambda)W_4+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_3\\
&\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_2\\
&\quad +(\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta)W_1-\alpha\beta\lambda\gamma\delta W_0.
\end{aligned}$$

and we assume that the roots $\alpha,\beta,\gamma,\delta,\lambda,\mu$ of the characteristic equation

$$x^6-rx^5-sx^4-tx^3-ux^2-vx-y=0 \tag{1.2}$$

of W_n are all distinct.

Note that we have the following identities

$$\left\{\begin{array}{c} \alpha+\beta+\gamma+\delta+\lambda+\mu=r, \\ \alpha\beta+\alpha\lambda+\alpha\gamma+\alpha\mu+\beta\lambda+\alpha\delta+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\beta\delta+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta=-s, \\ \alpha\beta\lambda+\alpha\beta\gamma+\alpha\beta\mu+\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\beta\delta+\alpha\lambda\delta+\alpha\gamma\mu+\beta\lambda\gamma+\beta\lambda\mu+\alpha\gamma\delta+\alpha\mu\delta+\beta\lambda\delta \\ +\beta\gamma\mu+\lambda\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta=t, \\ \alpha\beta\lambda\gamma+\alpha\beta\lambda\mu+\alpha\beta\lambda\delta+\alpha\beta\gamma\mu+\alpha\lambda\gamma\mu+\alpha\beta\gamma\delta+\alpha\beta\mu\delta+\alpha\lambda\gamma\delta+\alpha\lambda\mu\delta+\beta\lambda\gamma\mu+\alpha\gamma\mu\delta \\ +\beta\lambda\gamma\delta+\beta\lambda\mu\delta+\beta\gamma\mu\delta+\lambda\gamma\mu\delta=-u \\ \alpha\beta\lambda\gamma\mu+\alpha\beta\lambda\gamma\delta+\alpha\beta\lambda\mu\delta+\alpha\beta\gamma\mu\delta+\alpha\lambda\gamma\mu\delta+\beta\lambda\gamma\mu\delta=v, \\ \alpha\beta\lambda\gamma\mu\delta=-y. \end{array}\right. \tag{1.3}$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n. This result of Howard and Saidak [10] is even true in the case of higher-order recurrence relations as the following theorem shows.

Theorem 1.2 ([10]). *Let $\{w_n\}$ be a sequence such that*

$$\{w_n\}=a_1w_{n-1}+a_2w_{n-2}+...+a_kw_{n-k}$$

for all integers n, with arbitrary initial conditions $w_0, w_1, ..., w_{k-1}$. Assume that each a_i and the initial conditions are complex numbers. Write

$$\begin{aligned} f(x) &= x^k-a_1x^{k-1}-a_2x^{k-2}-...-a_{k-1}x-a_k \\ &= (x-\alpha_1)^{d_1}(x-\alpha_2)^{d_2}...(x-\alpha_h)^{d_h} \end{aligned} \tag{1.4}$$

with $d_1+d_2+...+d_h=k$, and $\alpha_1, \alpha_2, ..., \alpha_k$ distinct. Then

(a) *For all n,*

$$w_n=\sum_{m=1}^{k}N(n,m)(\alpha_m)^n \tag{1.5}$$

where

$$N(n,m)=A_1^{(m)}+A_2^{(m)}n+...+A_{r_m}^{(m)}n^{r_m-1}=\sum_{u=0}^{r_m-1}A_{u+1}^{(m)}n^u$$

with each $A_i^{(m)}$ a constant determined by the initial conditions for $\{w_n\}$. Here, equation (1.5) is called the Binet form (or Binet formula) for $\{w_n\}$. We assume that $f(0)\neq 0$ so that $\{w_n\}$ can be extended to negative integers n.

If the zeros of (1.4) are distinct, as they are in our examples, then

$$w_n=A_1(\alpha_1)^n+A_2(\alpha_2)^n+...+A_k(\alpha_k)^n.$$

(b) *The Binet form for $\{w_n\}$ is valid for all integers n.*

Now we define two special cases of the generalized (r,s,t,u,v,y) sequence $\{W_n\}$. (r,s,t,u,v,y) sequence $\{G_n\}_{n\geq 0}$ and Lucas (r,s,t,u,v,y) sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the sixth-order recurrence relations

$$\begin{aligned} G_{n+6} &= rG_{n+5}+sG_{n+4}+tG_{n+3}+uG_{n+2}+vG_{n+1}+yG_n, \\ G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s, G_4=r^3+2sr+t, G_5=r^4+s^2+3r^2s+2rt+u, \\ H_{n+6} &= rH_{n+5}+sH_{n+4}+tH_{n+3}+uH_{n+2}+vH_{n+1}+yH_n, \\ H_0 &= 6, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t, H_4=r^4+4r^2s+4tr+2s^2+4u, \\ H_5 &= r^5+5r^3s+5tr^2+5rs^2+5ur+5ts+5v, \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{v}{y}G_{-(n-1)} - \frac{u}{y}G_{-(n-2)} - \frac{t}{y}G_{-(n-3)} - \frac{s}{y}G_{-(n-4)} - \frac{r}{y}G_{-(n-5)} + \frac{1}{y}G_{-(n-6)}, \\ H_{-n} &= -\frac{v}{y}H_{-(n-1)} - \frac{u}{y}H_{-(n-2)} - \frac{t}{y}H_{-(n-3)} - \frac{s}{y}H_{-(n-4)} - \frac{r}{y}H_{-(n-5)} + \frac{1}{y}H_{-(n-6)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

Some special cases of (r, s, t, u, v) sequence $\{G_n(0, 1, r, r^2 + s, r^3 + 2sr + t; r, s, t, u, v)\}$ and Lucas (r, s, t, u, v) sequence $\{H_n(4, r, 2s + r^2, r^3 + 3sr + 3t, r^4 + 4r^2s + 4tr + 2s^2 + 4u; r, s, t, u, v)\}$ are as follows:Some special cases of (r, s, t, u, v, y) sequence $\{G_n(0, 1, r, r^2 + s, r^3 + 2sr + t, r^4 + s^2 + 3r^2s + 2rt + u; r, s, t, u, v, y)\}$ and Lucas (r, s, t, u, v, y) sequence $\{H_n(4, r, 2s + r^2, r^3 + 3sr + 3t, r^4 + 4r^2s + 4tr + 2s^2 + 4u, r^5 + 5r^3s + 5tr^2 + 5rs^2 + 5ur + 5ts + 5v; r, s, t, u, v, y)\}$ are as follows:

1. $G_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 1) = H_n$, Hexanacci sequence,
2. $H_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 1) = E_n$, Hexanacci-Lucas sequence,
3. $G_n(0, 1, 2, 5, 13, 34; 2, 1, 1, 1, 1, 1) = P_n$, sixth-order Pell sequence,
4. $H_n(6, 2, 6, 17, 46, 122; 2, 1, 1, 1, 1, 1) = Q_n$, sixth-order Pell-Lucas sequence.

For all integers n, (r, s, t, u, v, y), Lucas (r, s, t, u, v, y) and modified (r, s, t, u, v, y) numbers can be expressed, respectively, using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ &\quad + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)} + \frac{\delta^{n+4}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ &\quad + \frac{\lambda^{n+4}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)} + \frac{\mu^{n+4}}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}, \\ H_n &= \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n + \mu^n, \end{aligned}$$

respectively.

2 THE PROOF OF THEOREM 1.1

To prove Theorem 1.1, we need the following lemma.

Lemma 2.1. *For $n \in \mathbb{Z}$, denote*
$S_n = \alpha^n\beta^n\lambda^n\gamma^n\mu^n + \alpha^n\beta^n\lambda^n\gamma^n\delta^n + \alpha^n\beta^n\lambda^n\mu^n\delta^n + \alpha^n\beta^n\gamma^n\mu^n\delta^n + \alpha^n\lambda^n\gamma^n\mu^n\delta^n + \beta^n\lambda^n\gamma^n\mu^n\delta^n$
where $\alpha, \beta, \gamma, \delta, \lambda$ and μ are as in defined in Formula (1.3). Then the followings hold:

(a) *For $n \in \mathbb{Z}$, we have $S_n = (-y)^n H_{-n}$ and $S_{-n} = (-y)^{-n} H_n$.*

(b) *S_n has the recurrence relation so that*

$$S_n = vS_{n-1} - uyS_{n-2} + ty^2S_{n-3} - sy^3S_{n-4} + ry^4S_{n-5} + y^5S_{n-6}$$

with the initial conditions $S_0 = 6$, $S_1 = v$, $S_2 = v^2 - 2uy, S_3 = v^3 - 3uvy + 3ty^2, S_4 = 2u^2y^2 - 4uv^2y + v^4 + 4tvy^2 - 4sy^3, S_5 = 5u^2vy^2 - 5uv^3y - 5tuy^3 + v^5 + 5tv^2y^2 - 5svy^3 + 5ry^4$. The sequence at negative indices is given by

$$S_{-n} = -\frac{ry^4}{y^5}S_{-n+1} - \frac{-sy^3}{y^5}S_{-n+2} - \frac{ty^2}{y^5}S_{-n+3} - \frac{-uy}{y^5}S_{-n+4} - \frac{v}{y^5}S_{-n+5} + \frac{1}{y^5}S_{-n+6}, \text{ for } n = 1, 2, 3, ...$$

(c) S_n *has the identity so that*

$$S_n = \frac{1}{120}(H_n^5 - 10H_n^3 H_{2n} + 15H_{2n}^2 H_n + 20H_n^2 H_{3n} + 24H_{5n} - 30H_{4n}H_n - 20H_{3n}H_{2n}).$$

Proof.

(a) From the definition of S_n and H_n, we obtain

$$\begin{aligned} H_{-n} &= \alpha^{-n} + \beta^{-n} + \gamma^{-n} + \delta^{-n} + \lambda^{-n} + \mu^{-n} \\ &= \frac{\alpha^n\beta^n\lambda^n\gamma^n\mu^n + \alpha^n\beta^n\lambda^n\gamma^n\delta^n + \alpha^n\beta^n\lambda^n\mu^n\delta^n + \alpha^n\beta^n\gamma^n\mu^n\delta^n + \alpha^n\lambda^n\gamma^n\mu^n\delta^n + \beta^n\lambda^n\gamma^n\mu^n\delta^n}{\alpha^n\beta^n\lambda^n\gamma^n\mu^n\delta^n} \\ &= \frac{S_n}{(-y)^n}. \end{aligned}$$

i.e., $S_n = v^n H_{-n}$ and so $S_{-n} = v^{-n}H_n$.

(b) With Formula (1.3) or using the formula $S_n = v^n H_{-n}$, we obtain initial values of S_n as

$$\begin{aligned} S_0 &= (-y)^0 H_0 = 6, \\ S_1 &= (-y)^1 H_{-1} = (-y)^1(-\frac{v}{y}) = v. \\ S_2 &= (-y)^2 H_{-2} = (-y)^2(\frac{1}{y^2}(-2uy + v^2)) = v^2 - 2uy. \\ S_3 &= (-y)^3 H_{-3} = (-y)^3(-\frac{1}{y^3}(3ty^2 + v^3 - 3uvy)) = v^3 - 3uvy + 3ty^2, \\ S_4 &= (-y)^4 H_{-4} = (-y)^4(\frac{1}{y^4}(-4sy^3 + 2u^2y^2 + v^4 + 4tvy^2 - 4uv^2y)) \\ &= 2u^2y^2 - 4uv^2y + v^4 + 4tvy^2 - 4sy^3, \\ S_5 &= (-y)^5 H_{-5} = (-y)^5(-\frac{1}{y^5}(5ry^4 + v^5 - 5svy^3 - 5tuy^3 - 5uv^3y + 5tv^2y^2 + 5u^2vy^2)) \\ &= 5u^2vy^2 - 5uv^3y - 5tuy^3 + v^5 + 5tv^2y^2 - 5svy^3 + 5ry^4. \end{aligned}$$

For $n \geq 6$, we have

$$\begin{aligned} vS_{n-1} &= S_1 S_{n-1} \\ &= \alpha^n\beta^n\lambda^n\gamma^n\mu^n + \alpha^n\beta^n\lambda^n\gamma^n\delta^n + \alpha^n\beta^n\lambda^n\mu^n\delta^n + \alpha^n\beta^n\gamma^n\mu^n\delta^n + \alpha^n\lambda^n\gamma^n\mu^n\delta^n + \beta^n\lambda^n\gamma^n\mu^n\delta^n \\ &\quad +\alpha^{n-1}\beta^{n-1}\lambda^{n-1}\gamma^{n-1}\mu^{n-1}(\alpha\beta\lambda\gamma\delta + \alpha\beta\lambda\mu\delta + \alpha\beta\gamma\mu\delta + \alpha\lambda\gamma\mu\delta + \beta\lambda\gamma\mu\delta) \\ &\quad +\alpha^{n-1}\beta^{n-1}\lambda^{n-1}\gamma^{n-1}\delta^{n-1}(\alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\mu\delta + \alpha\beta\gamma\mu\delta + \alpha\lambda\gamma\mu\delta + \beta\lambda\gamma\mu\delta) \\ &\quad +\alpha^{n-1}\beta^{n-1}\lambda^{n-1}\mu^{n-1}\delta^{n-1}(\alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\gamma\delta + \alpha\beta\gamma\mu\delta + \alpha\lambda\gamma\mu\delta + \beta\lambda\gamma\mu\delta) \\ &\quad +\alpha^{n-1}\beta^{n-1}\gamma^{n-1}\mu^{n-1}\delta^{n-1}(\alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\gamma\delta + \alpha\beta\lambda\mu\delta + \alpha\lambda\gamma\mu\delta + \beta\lambda\gamma\mu\delta) \\ &\quad +\alpha^{n-1}\lambda^{n-1}\gamma^{n-1}\mu^{n-1}\delta^{n-1}(\alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\gamma\delta + \alpha\beta\lambda\mu\delta + \alpha\beta\gamma\mu\delta + \beta\lambda\gamma\mu\delta) \\ &\quad +\beta^{n-1}\lambda^{n-1}\gamma^{n-1}\mu^{n-1}\delta^{n-1}(\alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\gamma\delta + \alpha\beta\lambda\mu\delta + \alpha\beta\gamma\mu\delta + \alpha\lambda\gamma\mu\delta) \\ &= -S_{n-6}y^5 - rS_{n-5}y^4 + sS_{n-4}y^3 - tS_{n-3}y^2 + uS_{n-2}y + S_n \end{aligned}$$

and so

$$S_n = vS_{n-1} - uyS_{n-2} + ty^2S_{n-3} - sy^3S_{n-4} + ry^4S_{n-5} + y^5S_{n-6}.$$

(c) From the definition of S_n and H_n, we get

$(\alpha^n\beta^n + \alpha^n\lambda^n + \alpha^n\gamma^n + \alpha^n\mu^n + \beta^n\lambda^n + \alpha^n\delta^n + \beta^n\gamma^n + \beta^n\mu^n + \lambda^n\gamma^n + \lambda^n\mu^n + \beta^n\delta^n + \lambda^n\delta^n + \gamma^n\mu^n + \gamma^n\delta^n + \mu^n\delta^n) = \frac{1}{2}(H_n^2 - H_{2n})$,

$(\alpha^{2n}\beta^{2n} + \alpha^{2n}\lambda^{2n} + \alpha^{2n}\gamma^{2n} + \alpha^{2n}\mu^{2n} + \beta^{2n}\lambda^{2n} + \alpha^{2n}\delta^{2n} + \beta^{2n}\gamma^{2n} + \beta^{2n}\mu^{2n} + \lambda^{2n}\gamma^{2n} + \lambda^{2n}\mu^{2n} + \beta^{2n}\delta^{2n} + \lambda^{2n}\delta^{2n} + \gamma^{2n}\mu^{2n} + \gamma^{2n}\delta^{2n} + \mu^{2n}\delta^{2n}) = \frac{1}{2}(H_{2n}^2 - H_{4n})$,

$(\alpha^n\beta^n\lambda^n + \alpha^n\beta^n\gamma^n + \alpha^n\beta^n\mu^n + \alpha^n\lambda^n\gamma^n + \alpha^n\lambda^n\mu^n + \alpha^n\beta^n\delta^n + \alpha^n\lambda^n\delta^n + \alpha^n\gamma^n\mu^n + \beta^n\lambda^n\gamma^n + \beta^n\lambda^n\mu^n + \alpha^n\gamma^n\delta^n + \alpha^n\mu^n\delta^n + \beta^n\lambda^n\delta^n + \beta^n\gamma^n\mu^n + \lambda^n\gamma^n\mu^n + \beta^n\gamma^n\delta^n + \beta^n\mu^n\delta^n + \lambda^n\gamma^n\delta^n +$

$\lambda^n\mu^n\delta^n + \gamma^n\mu^n\delta^n) = \frac{1}{6}(-3H_nH_{2n} + H_n^3 + 2H_{3n})$.

It now follows that

$$\begin{aligned} H_n^5 &= H_{5n} + 5(H_{4n}H_n - H_{5n}) + 10\,(H_{3n}H_{2n} - H_{5n}) \\ &\quad +10(-2H_nH_{4n} + 2H_{5n} + H_n^2H_{3n} - H_{2n}H_{3n}) \\ &\quad +30(H_n\frac{1}{2}(H_{2n}^2 - H_{4n}) - H_{3n}H_{2n} + H_{5n}) \\ &\quad +10(6H_nH_{4n} - 6H_{5n} - 3H_nH_{2n}^2 - 3H_n^2H_{3n} + H_n^3H_{2n} + 5H_{2n}H_{3n}) \\ &\quad +120S_n \\ &\Rightarrow \\ S_n &= \frac{1}{120}(H_n^5 - 10H_n^3H_{2n} + 15H_{2n}^2H_n + 20H_n^2H_{3n} + 24H_{5n} - 30H_{4n}H_n - 20H_{3n}H_{2n}). \ \square \end{aligned}$$

Now, we shall complete the proof of Theorem 1.1.

The Proof of Theorem 1.1:

Note that for $n \in \mathbb{Z}$, we have the following:

$\alpha^n\beta^n + \alpha^n\lambda^n + \alpha^n\gamma^n + \alpha^n\mu^n + \beta^n\lambda^n + \alpha^n\delta^n + \beta^n\gamma^n + \beta^n\mu^n + \lambda^n\gamma^n + \lambda^n\mu^n + \beta^n\delta^n + \lambda^n\delta^n + \gamma^n\mu^n + \gamma^n\delta^n + \mu^n\delta^n = \frac{1}{2}(H_n^2 - H_{2n})$,

$\alpha^{2n}\beta^{2n} + \alpha^{2n}\lambda^{2n} + \alpha^{2n}\gamma^{2n} + \alpha^{2n}\mu^{2n} + \beta^{2n}\lambda^{2n} + \alpha^{2n}\delta^{2n} + \beta^{2n}\gamma^{2n} + \beta^{2n}\mu^{2n} + \lambda^{2n}\gamma^{2n} + \lambda^{2n}\mu^{2n} + \beta^{2n}\delta^{2n} + \lambda^{2n}\delta^{2n} + \gamma^{2n}\mu^{2n} + \gamma^{2n}\delta^{2n} + \mu^{2n}\delta^{2n} = \frac{1}{2}(H_{2n}^2 - H_{4n})$,

$\alpha^n\beta^n\lambda^n + \alpha^n\beta^n\gamma^n + \alpha^n\beta^n\mu^n + \alpha^n\lambda^n\gamma^n + \alpha^n\lambda^n\mu^n + \alpha^n\beta^n\delta^n + \alpha^n\lambda^n\delta^n + \alpha^n\gamma^n\mu^n + \beta^n\lambda^n\gamma^n + \beta^n\lambda^n\mu^n + \alpha^n\gamma^n\delta^n + \alpha^n\mu^n\delta^n + \beta^n\lambda^n\delta^n + \beta^n\gamma^n\mu^n + \lambda^n\gamma^n\mu^n + \beta^n\gamma^n\delta^n + \beta^n\mu^n\delta^n + \lambda^n\gamma^n\delta^n + \lambda^n\mu^n\delta^n + \gamma^n\mu^n\delta^n = \frac{1}{6}(-3H_nH_{2n} + H_n^3 + 2H_{3n})$,

$\alpha^n\beta^n\lambda^n\gamma^n\mu^nA_4 + \alpha^n\beta^n\lambda^n\gamma^n\delta^nA_6 + \alpha^n\beta^n\lambda^n\mu^n\delta^nA_3 + \alpha^n\beta^n\gamma^n\mu^n\delta^nA_5 + \alpha^n\lambda^n\gamma^n\mu^n\delta^nA_2 + \beta^n\lambda^n\gamma^n\mu^n\delta^nA_1 = (-y)^nW_{-n}$,

$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = W_0$.

Now, for $n \in \mathbb{Z}$, we obtain

$$\begin{aligned} & W_n \times \frac{1}{24}(8H_nH_{3n} + H_n^4 - 6H_{4n} - 6H_n^2H_{2n} + 3H_{2n}^2) \\ =\ & (A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n + A_6\mu^n) \\ & (\alpha^n\beta^n\lambda^n\gamma^n + \alpha^n\beta^n\lambda^n\mu^n + \alpha^n\beta^n\lambda^n\delta^n + \alpha^n\beta^n\gamma^n\mu^n \\ & +\alpha^n\lambda^n\gamma^n\mu^n + \alpha^n\beta^n\gamma^n\delta^n + \alpha^n\beta^n\mu^n\delta^n + \alpha^n\lambda^n\gamma^n\delta^n \\ & +\alpha^n\lambda^n\mu^n\delta^n + \beta^n\lambda^n\gamma^n\mu^n + \alpha^n\gamma^n\mu^n\delta^n + \beta^n\lambda^n\gamma^n\delta^n \\ & +\beta^n\lambda^n\mu^n\delta^n + \beta^n\gamma^n\mu^n\delta^n + \lambda^n\gamma^n\mu^n\delta^n) \\ =\ & \frac{1}{6}(6H_nW_{4n} - 6W_{5n} - 3H_n^2W_{3n} + H_n^3W_{2n} + 3H_{2n}W_{3n} \\ & +2H_{3n}W_{2n} - 3H_nH_{2n}W_{2n}) + (S_nW_0 - (-y)^nW_{-n}). \end{aligned}$$

By Lemma 2.1 (c) (using $S_n = \frac{1}{120}(H_n^5 - 10H_n^3H_{2n} + 15H_{2n}^2H_n + 20H_n^2H_{3n} + 24H_{5n} - 30H_{4n}H_n - 20H_{3n}H_{2n})$), it follows that

$W_{-n} = \frac{1}{120}(-y)^{-n}(W_0H_n^5 - 5W_nH_n^4 + 20H_n^3W_{2n} - 10W_0H_n^3H_{2n} + 15W_0H_{2n}^2H_n - 15W_nH_{2n}^2 + 20W_0H_n^2H_{3n} + 30W_nH_n^2H_{2n} - 60H_n^2W_{3n} - 120W_{5n} + 120H_nW_{4n} + 60H_{2n}W_{3n} + 40H_{3n}W_{2n} + 24W_0H_{5n} + 30H_{4n}W_n - 20W_0H_{3n}H_{2n} - 30H_{4n}H_nW_0 - 40H_{3n}H_nW_n - 60H_{2n}H_nW_{2n})$

$= (-1)^{-n-1}y^{-n}(W_{5n} - H_nW_{4n} + \frac{1}{2}(H_n^2 - H_{2n})W_{3n} - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_{2n} + \frac{1}{24}(H_n^4 +$

$3H_{2n}^2 - 6H_n^2H_{2n} - 6H_{4n} + 8H_{3n}H_n)W_n - \frac{1}{120}(H_n^5 - 10H_n^3H_{2n} + 15H_{2n}^2H_n + 20H_n^2H_{3n} + 24H_{5n} - 30H_{4n}H_n - 20H_{3n}H_{2n})W_0)$. $\square$

Next, we present a remark which presents how H_n can be written in terms of W_n.

Remark 2.1. To express W_{-n} by the sequence itself at positive indices we need that H_n can be written in terms of W_n. For this, writing

$$H_n = a \times W_{n+5} + b \times W_{n+4} + c \times W_{n+3} + d \times W_{n+2} + e \times W_{n+1} + f \times W_n$$

and solving the system of equations

$$\begin{aligned}
H_0 &= a \times W_5 + b \times W_4 + c \times W_3 + d \times W_2 + e \times W_1 + f \times W_0 \\
H_1 &= a \times W_6 + b \times W_5 + c \times W_4 + d \times W_3 + e \times W_2 + f \times W_1 \\
H_2 &= a \times W_7 + b \times W_6 + c \times W_5 + d \times W_4 + e \times W_3 + f \times W_2 \\
H_3 &= a \times W_8 + b \times W_7 + c \times W_6 + d \times W_5 + e \times W_4 + f \times W_3 \\
H_4 &= a \times W_9 + b \times W_8 + c \times W_7 + d \times W_6 + e \times W_5 + f \times W_4 \\
H_5 &= a \times W_{10} + b \times W_9 + c \times W_8 + d \times W_7 + e \times W_6 + f \times W_5
\end{aligned}$$

or

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} W_5 & W_4 & W_3 & W_2 & W_1 & W_0 \\ W_6 & W_5 & W_4 & W_3 & W_2 & W_1 \\ W_7 & W_6 & W_5 & W_4 & W_3 & W_2 \\ W_8 & W_7 & W_6 & W_5 & W_4 & W_3 \\ W_9 & W_8 & W_7 & W_6 & W_5 & W_4 \\ W_{10} & W_9 & W_8 & W_7 & W_6 & W_5 \end{pmatrix}^{-1} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{pmatrix}$$

we find a, b, c, d, e, f so that H_n can be written in terms of W_n and we can replace this H_n in Theorem 1.1.

Using Theorem 1.1, we have the following corollary.

Corollary 2.2. *For $n \in \mathbb{Z}$, we have*

$H_{-n} = \frac{1}{120}(-y)^{-n}(H_n^5 + 15H_{2n}^2H_n + 20H_n^2H_{3n} - 10H_n^3H_{2n} + 24H_{5n} - 30H_{4n}H_n - 20H_{3n}H_{2n})$

Using Theorem 1.1 and Remark 2.1 (or the last corollary), we can give some formulas for the special cases of generalized Hexanacci sequence (generalized (r, s, t, u, v, y)-sequence) as follows.

We have the following corollary which gives the connection between the special cases of generalized Hexanacci sequence at the positive index and the negative index.

Corollary 2.3. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Hexanacci-Lucas sequence:*
$E_{-n} = \frac{1}{120(-1)^n}(E_n^5 - 10E_n^3E_{2n} + 15E_{2n}^2E_n + 20E_n^2E_{3n} + 24E_{5n} - 30E_{4n}E_n - 20E_{3n}E_{2n})$.

(b) *sixth order Pell-Lucas sequence:*
$Q_{-n} = \frac{1}{120(-1)^n}(Q_n^5 - 10Q_n^3Q_{2n} + 15Q_{2n}^2Q_n + 20Q_n^2Q_{3n} + 24Q_{5n} - 30Q_{4n}Q_n - 20Q_{3n}Q_{2n})$.

(c) *modified sixth-order Jacobsthal-Lucas sequence:*
$R_{-n} = \frac{1}{120(-2)^n}(R_n^5 - 10R_n^3R_{2n} + 15R_{2n}^2R_n + 20R_n^2R_{3n} + 24R_{5n} - 30R_{4n}R_n - 20R_{3n}R_{2n})$.

(d) *Lucas 6-primes sequence:*
$H_{-n} = \frac{1}{120(-13)^n}(H_n^5 - 10H_n^3H_{2n} + 15H_{2n}^2H_n + 20H_n^2H_{3n} + 24H_{5n} - 30H_{4n}H_n - 20H_{3n}H_{2n})$.

3 CONCLUSION

The main results of this paper propose new recurrence properties of generalized Hexanacci sequence. We consider generalized Hexanacci sequence at negative indices and construct the relationship between the sequence and itself at positive indices. This illustrates the recurrence property of the sequence at the negative index. Meanwhile, this connection holds for all integers.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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