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Linearized Oscillations in Autonomous Delay Impulsive Differential Equations

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Abstract

In recent years, there have been intensive efforts to establish linearised oscillation results for one-dimensional delay, neutral delay and advanced impulsive differential equations. An impressive number of these efforts have yielded fruitful results in many analytical and applied areas. This is particularly obvious in the areas of applied disciplines such as the linear delay impulsive differential equations. However, there still remains a lot more to be explored in this direction, especially, in the area of non-linear autonomous differential equations. In this paper, we are proposing the development of linearised oscillation techniques for some general non-linear autonomous impulsive differential equations with several delays.

Keywords: Linearised oscillation; Delay impulsive differential equations.

1 Introduction and Statement of the Problem

Presently, a linearised oscillation theory from which the investigation of the oscillatory behaviour of the solutions of certain class of non-linear impulsive differential equations can be reduced to that of the associated linear impulsive differential equations is being intensively investigated and developed. Precisely, the benefits of such research efforts are already being harnessed in one-dimensional non-linear impulsive differential equations with single variable and constant delays ([1]).

In this paper, we are proposing the establishment of linearised oscillation techniques for some general type of non-linear autonomous delay impulsive differential equations defined below.

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Notation 1.1. *Let $J = (\alpha, \beta) \subset R, \ -\infty \leq \alpha < \beta \leq \infty$ denote the domain of investigation.*

Definition 1.1. Let $S := \{t_k\}_{k \in E} \subset J$ be a strictly ascending sequence of the time moments of impulse effects and let E be a subscript set which can be the set of natural numbers $\mathbb{N}$ or the set of integers Z such that

- $t_k \to \infty$ if $k \to \infty$ and if $E = Z$ then $t_k \to -\infty$ if $k \to -\infty$;
- $t_k \geq 0$ if $k \geq 0$.

Our equation then has the form

$$\begin{cases} x^{'}(t) + f\left(t, x\left(t-\tau_1\right), \cdots, \ x\left(t-\tau_m\right)\right) = 0, \ \ t \in J \setminus S; \\ \Delta x\left(t_k\right) + g\left(t_k, \ x\left(t_k-\tau_1\right), \cdots, \ x\left(t_k-\tau_m\right)\right) = 0 \ t_k \in S, \end{cases} \tag{1.1}$$

where $\tau_i > 0, \ \ 1 \leq i \leq m$, are constants. This means that the functions f and g are non-linear and depend, among others, on several constant delays.

In order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows: Let $D := [T, \bar{T}) \subset J \subset R$ and let the set of impulse points S be fixed.

Definition 1.2. Let
$PC\left(D, R\right) := \{\varphi \mid \varphi : D \to R, \ \varphi \in C(D \setminus S), \ \exists \, \varphi(t-0), \varphi(t+0), \ \forall t \in D\}.$

From the studies in Bainov and Simeonov (1998), Lakshmikantham et al. (1989) and Isaac et al. (2011) ([1], [4], [2]) we define the function space $\forall \ r \in \mathbb{N}$:

Definition 1.3. $PC^r\left(D, R\right) := \{\varphi \mid \varphi \in PC\left(D, R\right), \dfrac{d^j\varphi}{dt^j} \in PC\left(D, R\right)), \ \forall \ 1 \leq j \leq r, \}.$

To specify the points of discontinuity of functions belonging to PC or PC^r, we shall sometimes use the symbols $PC(D, R; S)$ and $PC^r(D, R; S)$, $r \in N$.

For easy reference in the study, we introduce the following conditions:

$$\textbf{C.1.1} \begin{cases} f \in C\left[J \times R^m, \ R\right], \ g \in C\left[S \times R^m, \ R\right], \quad \text{and } \forall t \in J \setminus S, \ \forall t_k \in S, \\ f\left(t, u_1, \cdots, u_m\right) \geq 0, \ \ g\left(t_k, u_1, \cdots, u_m\right) \geq 0 \ \ \text{ if } \ \ u_i \geq 0, \ \ 1 \leq i \leq m; \\ f\left(t, u_1, \cdots, u_m\right) \leq 0, \ \ g\left(t_k, u_1, \cdots, u_m\right) \leq 0 \ \ \text{if } \ u_i \leq 0, \ \ 1 \leq i \leq m. \end{cases}$$

There exist $T_p \geq t_0$ and $\delta > 0$, such that $\forall t \in J \setminus S, \ t \geq T_p$ and $\forall k \in \mathbb{N}, \ t_k \in S, \ t_k \geq T_p$,

$$\textbf{C.1.2} \begin{cases} f\left(t, u_1, \cdots, u_m\right) \ \text{ and } \ g\left(t_k, u_1, \cdots, u_m\right) \ \text{ are increasing in } u_1, \ \cdots, \ u_m \\ \text{in the sense that if } 0 \leq u_i^{'} \leq u_i^{''} \leq \delta, \ \ 1 \leq i \leq m, \text{ then} \\ f\left(t, u_1^{'}, \cdots, u_m^{'}\right) \leq \ f\left(t, u_1^{''}, \cdots, u_m^{''}\right) \text{ and} \\ g\left(t_k, u_1^{'}, \cdots, u_m^{'}\right) \leq g\left(t_k, u_1^{''}, \cdots, u_m^{''}\right). \end{cases}$$

$$\textbf{C.1.3} \begin{cases} f\left(t, u_1, \cdots, u_m\right) \geq \sum\limits_{i=1}^{m} p_i\left(t\right) u_i > 0; \quad g\left(t_k, u_1, \cdots, u_m\right) \geq \sum\limits_{i=1}^{m} p_{i0} \times u_i > 0 \\ \text{for } 0 < u_1, \ \cdots, \ u_m \leq \delta \text{ and} \\ f\left(t, u_1, \cdots, u_m\right) \leq \sum\limits_{i=1}^{m} p_i\left(t\right) u_i < 0; \ \ g\left(t_k, u_1, \cdots, u_m\right) \leq \sum\limits_{i=1}^{m} p_{i0} \times u_i < 0 \\ \text{for } 0 > u_1, \ \cdots, \ u_m \geq -\delta. \end{cases}$$

or

$$\textbf{C.1.4}\begin{cases}\lim\limits_{\substack{(u_1,\cdots,u_m)\to 0\\ u_i\times u_j>0,\ 1\le i,j\le m}}\dfrac{f(t,u_1,\cdots,u_m)}{\sum\limits_{q=1}^{m}p_q(t)\times u_q}\equiv 1 \text{ and}\\ \lim\limits_{\substack{(u_1,\cdots,u_m)\to 0\\ u_i\times u_j>0,\ 1\le i,j\le m}}\dfrac{g(t_k,u_1,\cdots,u_m)}{\sum\limits_{q=1}^{m}p_{q,0}\times u_q}\equiv 1,\end{cases}$$

where $p_1,\ \ p_2,\ \cdots,\ p_m \in$ PC $\left[[t_0,\ \infty),\ R^+\right]$ and $p_{1,0},\ \ p_{2,0},\ \cdots,\ p_{m,0}\in R^+$. Similar to the stability theory of differential and difference equations ([3], [4], [5]), we shall prove that, with appropriate hypotheses, the oscillatory behaviour of the non-linear autonomous delay impulsive differential equation (1.1) is characterized by the oscillatory behaviour of the associated linear differential equation

$$\begin{cases}y'(t)+\sum\limits_{i=1}^{m}p_i(t)\,y(t-\tau_i)=0,\ \ t\in J\setminus S\\ \Delta y(t_k)+\sum\limits_{i=1}^{m}p_{i,0}\times y(t_k-\tau_i)=0,\ t_k\in S,\end{cases}\tag{1.2}$$

where $p_i(t),\, t\ge t_0$ and $p_{i,0}$ are as defined in conditions C.1.3 and C.1.4.

Recall that in a one-dimensional set up, a non-trivial solution $y(t)$ of a delay impulsive differential equation is said to be oscillatory if it is possible to choose $T\ge t_0$ such that for $t>T$, $y(t)$ is neither finally positive nor finally negative, where $y(t)$ is

(i) finally positive if there exists $T\ge 0$ such that $y(t)$ is defined for $t\ge T$ and $y(t)>0$ for $t\ge T$;

(ii) finally negative if there exists $T\ge 0$ such that $y(t)$ is defined for $t\ge T$ and $y(t)<0$ for $t\ge T$ ([1], [4], [2]).

All functional inequalities proposed are assumed to hold finally, that is, for all sufficiently large t.

2 Main Results

In this section, we shall state and prove the related oscillatory theorems, but will first establish some fundamental lemmas.

Lemma 2.1. *Suppose that* $m\ge 1$ *and let condition C.1.1 hold. Then every non-oscillatory solution of equation* (1.1) *tends to zero as* $t\to\infty$.

Proof. Let $x(t)$ be a non-oscillatory solution of equation (1.1). We assume that $x(t)$ is finally positive. The case where $x(t)$ is finally negative is similar and will be omitted. Then finally,

$$\begin{cases}x'(t)=-f(t,x(t-\tau_1),\cdots,\ x(t-\tau_m))\le 0,\ \ t\notin S\\ \Delta x(t_k)=-g(t_k,\ x(t_k-\tau_1),\cdots,\ x(t_k-\tau_m))\le 0,\ \forall t_k\in S.\end{cases}\tag{2.1}$$

Choose $T\ge t_0$ such that $x(t-\tau_i)>0$ for $t>T$ and $1\le i\le m$. From equation (2.1), it is clear that $x(t)$ is decreasing for $t\ge T$ and so

$$L=\lim_{t\to\infty}x(t)$$

exists and $L\ge 0$.

To complete the proof, we must show that $L=0$. For the sake of contradiction, let us assume that $L>0$. Then integrating both sides of equation (1.1) from t_0 to ∞, we obtain

$$L = x(t_0) - \left[\int_{t_0}^{\infty} f\left(s, x\left(s-\tau_1\right), \cdots, x\left(s-\tau_m\right)\right) \mathrm{ds} - \right.$$
$$\left. - \sum_{t_0 \le t_k < \infty} g\left(t_k, x\left(t_k-\tau_1\right), \cdots, x\left(t_k-\tau_m\right)\right) \right]$$

which implies $\lim_{t\to\infty} x(t) = -\infty$ since $M + M_0 < 0$, where

$$M = \int_{t_0}^{\infty} f\left(s, x\left(s-\tau_1\right), \cdots, x\left(s-\tau_m\right)\right) \mathrm{ds}$$

and

$$M_0 = \sum_{t_0 \le t_k < \infty} g\left(t_k, x\left(t_k-\tau_1\right), \cdots, x\left(t_k-\tau_m\right)\right).$$

This is a contradiction and hence $L = 0$. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let conditions C.1.1 and C.1.2 be fulfilled. Assume that for $t \ge T_p$ the inequality*

$$\begin{cases} x'(t) + f\left(t, x\left(t-\tau_1\right), \cdots,\ x\left(t-\tau_m\right)\right) \le 0, \ \ t \notin S \\ \Delta x\left(t_k\right) + g\left(t_k,\ x\left(t_k-\tau_1\right), \cdots,\ x\left(t_k-\tau_m\right)\right) \le 0, \ \forall t_k \in S \end{cases} \tag{2.2}$$

has a finally positive solution $\bar{x}(t)$ with $\bar{x}(t) \le \delta$. Then equation (1.1) *has a finally positive solution $x(t)$ with $x(t) \le \bar{x}(t)$ for t sufficiently large.*

Proof. We begin by using assumption C.1.2 and the condition of the lemma ($\bar{x}(t) \le \delta,\ T_p \le t \in J$) that there is a T_p such that for $T_p \le T_{-1} \le t \in J$, $0 \le \bar{x}(t-\tau_i) \le \delta$ and $1 \le i \le m$, where T_{-1} is defined by

$$T_{-1} = t_0 - \max\left\{\tau_1,\ \tau_2,\ \cdots,\ \tau_m\right\}. \tag{2.3}$$

It then follows that

$$\bar{x}(t) \text{ is strictly decreasing for } t \ge T_{-1}. \tag{2.4}$$

Consequently, $\lim_{t\to\infty} \bar{x}(t) = L \in [0,\ \infty)$ exists. We integrate both sides of inequality (2.2) from t to ∞ and obtain

$$L + \int_{t}^{\infty} f\left(s, \bar{x}\left(s-\tau_1\right), \cdots, \bar{x}\left(s-\tau_m\right)\right) \mathrm{ds} +$$
$$+ \sum_{t \le t_k < \infty} g\left(t_k, \bar{x}\left(t_k-\tau_1\right), \cdots, \bar{x}\left(t_k-\tau_m\right)\right) \le \bar{x}(t),\ t \ge T_p. \tag{2.5}$$

Now, we define the set of functions

$$\theta = \left\{\mu \in PC\left[\left[T_{-1},\ \infty\right) \setminus S,\ [0,\ \infty)\right] : 0 \le \mu(t) \le \bar{x}(t) \text{ for } t \ge T_{-1}\right\}$$

and define operator H on θ as

$$(H\mu)(t) = L + \int_{t}^{\infty} F(s,\ \mu)\, \mathrm{ds}\ + \sum_{t \le t_k < \infty} G\left(t_k,\ \mu\right) \ \text{ for }\ t \ge T_{-1}, \tag{2.6}$$

where

$$F(s,\ \mu) = f(s,\ \mu(s-\tau_1),\ \cdots,\ \mu(s-\tau_m)),\ \forall s \in [T_{-1}, \infty) \setminus S \text{ and}$$
$$G(t_k, \mu) = g(t_k,\ \mu(t_k-\tau_1),\ \cdots, \mu(t_k-\tau_m)),\ \forall t_k \in S.$$

Clearly, $(H\mu) \in PC[[T_{-1},\ \infty) \setminus S,\ [0,\ \infty)]$. In view of C.1.2, $\mu_1,\ \mu_2 \in \theta$ with $\mu_1 \le \mu_2$ implies that $H\mu_1 \le H\mu_2$. Note also that from the inequality (2.5), $H\bar{x} \le \bar{x}$. Hence, $\mu \in \theta$ implies that $H\mu \le H\bar{x} \le \bar{x}$ and so, we see that $H : \theta \to \theta$. Next, we define the following sequence in θ:

$$x_0 = \bar{x} \text{ and } x_n = Hx_{n-1} \text{ for } n = 1,\ 2,\ \cdots.$$

It is clear, by induction, that

$$0 \le x_n(t) \le x_{n-1}(t) \le \bar{x}(t),\ t \ge T_{-1}. \tag{2.7}$$

Set

$$x(t) = \lim_{n\to\infty} x_n(t),\ t \ge T_{-1}. \tag{2.8}$$

Then, by the continuity of the functions $f,\ g$, we see that for $t,\ t_k \ge T_{-1}$,

$$\lim_{n\to\infty} f(t,\ x_n(t-\tau_1),\ \cdots,\ x_n(t-\tau_m)) = f(t,\ x(t-\tau_1),\ \cdots,\ x(t-\tau_m))$$
$$\lim_{n\to\infty} g(t_k,\ x_n(t_k-\tau_1),\ \cdots,\ x_n(t_k-\tau_m)) = g(t_k,\ x(t_k-\tau_1),\ \cdots,\ x(t_k-\tau_m)),$$
$$\forall k \in \mathbb{N}.$$

Also, observe that for $t \ge T_{-1}$,

$$f(t,\ x(t-\tau_1),\ \cdots,\ x(t-\tau_m)) \le f(t,\ \bar{x}(t-\tau_1),\ \cdots,\ \bar{x}(t-\tau_m))$$

and

$$g(t_k,\ x(t_k-\tau_1),\ \cdots,\ x(t_k-\tau_m)) \le g(t_k,\ \bar{x}(t_k-\tau_1),\ \cdots,\ \bar{x}(t_k-\tau_m)),$$
$$\forall k \in \mathbb{N}$$

what is more,

$$f(\cdot,\ \bar{x}(\cdot-\tau_1),\ \cdots,\ \bar{x}(\cdot-\tau_m)) \in L_1[T_{-1},\ \infty),$$
$$\{g(t_k,\ \bar{x}(t_k-\tau_1),\ \cdots,\ \bar{x}(t_k-\tau_m))\}_{k\in\mathbb{N}} \in l_1.$$

Therefore, it follows from Lebesgue's convergence theorem that $x(t)$ satisfies

$$x(t) = L + \int_t^\infty F(s,\ x)\,\mathrm{ds} + \sum_{t \le t_k < \infty} G(t_k,\ x) \quad \text{for } t \ge T_{-1}, \tag{2.9}$$

where

$$F(s,\ x) = f(s,\ x(s-\tau_1),\ \cdots,\ x(s-\tau_m)) \text{ and }, \forall s \in [T_{-1}, \infty) \setminus S;$$
$$G(t_k,\ x) = g(t_k,\ x(t_k-\tau_1),\ \cdots, x(t_k-\tau_m))\, \forall t_k \in S.$$

From equation (2.9), it is clear that $x(t) \in PC[[T_{-1},\ \infty], R]$ and piecewise continuously differentiable on $[T_{-1},\ \infty)$. Now, we claim that $x(t)$ is positive in $[T,\ \infty)$ for sufficiently large $T > T_{-1}$. In fact, if $T_{-1} \le t < T$, by condition (2.4) and the definition of $x_n(t)$, it is discovered that

$$0 \le \bar{x}(t) - \bar{x}(T) \le x_n(t)$$

which implies that

$$0 \le \bar{x}(t) - \bar{x}(T) \le x(t).$$

Next, we show that $x(t)$ is also positive in $[T,\ \infty)$. First assume that $L > 0$. Then from equation (2.9), it is obvious that $x(t) > 0$. Now, assume that $L = 0$. Let $\bar{t} = \inf\ \{t \geq T : x(t) > 0\}$. We claim that $\bar{t} = \infty$. Otherwise $\bar{t} \in [T,\ \infty)$. Hence $x(t) > 0$ for $T_{-1} \leq t < \bar{t}$ and $\bar{x}\left(\bar{t}\right) = 0$. But from condition $C.1.1$ and equation (2.9), we see that

$$x(\bar{t}) = \int_{\bar{t}}^{\infty} f\left(s, x\left(s-\tau_1\right), \cdots, x\left(s-\tau_m\right)\right) \mathrm{ds} + $$

$$+ \sum_{\bar{t} \leq t_k < \infty} g\left(t_k, x\left(t_k-\tau_1\right), \cdots, x\left(t_k-\tau_m\right)\right) > 0$$

and this contradicts our hypothesis and hence establishes our claim. Now, by differentiating both sides of equation (2.9), we obtain

$$\begin{cases} x^{'}(t) + f\left(t, x\left(t-\tau_1\right), \cdots,\ x\left(t-\tau_m\right)\right) \leq 0, \ \ t \notin S \\ \Delta x\left(t_k\right) + g\left(t_k,\ x\left(t_k-\tau_1\right), \cdots,\ x\left(t_k-\tau_m\right)\right) \leq 0,\ \forall t_k \in S \end{cases}$$

which implies that $x(t)$ is a positive solution of equation (2.9). Finally, from conditions (2.7) and (2.8), we see that $x\left(t\right) \leq \bar{x}\left(t\right)$. This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let us consider the linear autonomous delay differential equations*

$$\begin{cases} z^{'}(t) + \sum\limits_{i=1}^{m} a_i(t) z(t-\lambda_i) = 0,\ t \notin S \\ \Delta z(t_k) + \sum\limits_{i=1}^{m} a_{i,0} z(t_k-\lambda_i) = 0,\ \forall t_k \in S \end{cases} \tag{2.10}$$

and

$$\begin{cases} w^{'}(t) + \sum\limits_{i=1}^{m} b_i(t) w(t-\sigma_i) = 0,\ t \notin S \\ \Delta w(t_k) + \sum\limits_{i=1}^{m} b_{i,0} w(t_k-\sigma_i) = 0,\ \forall t_k \in S, \end{cases} \tag{2.11}$$

where, for each $1 \leq i \leq m$*,*

$$\begin{cases} a_i,\ b_i \in PC\left[\left[t_0,\ \infty\right),\ \left[0,\ \infty\right)\right],\ \textit{and}\ \lambda_i,\ \sigma_i,\ a_{i,0},\ b_{i,0} \in [0,\ \infty)\,; \\ \lim\limits_{t\to\infty} |t-\lambda_i| = \infty,\ \ \lim\limits_{t\to\infty} |t-\sigma_i| = \infty. \end{cases}$$

Assume that for each $1 \leq i \leq m$*, the functions* a_i *and* b_i *have the same set of zeros* ξ_i *with multiplicities and*

$$\lim_{\substack{(t\to\infty \\ t\neq\xi_i}} \frac{a_i(t)}{b_i(t)} = 1.$$

Then every solution of equation (2.10) *oscillates if and only if every solution of equation* (2.11) *oscillates.*

Theorem 2.1. *Assume that conditions* $C.1.1$ *to* $C.1.4$ *are satisfied and suppose that every solution of the linearised equation* (1.2) *is oscillatory. Then every solution of equation* (1.1) *also oscillates.*

Proof. Let us assume, for the sake of contradiction, that equation (1.1) has a non-oscillating solution $x(t)$. We assume that $x(t)$ is finally positive. The case where $x(t)$ is finally negative is similar and will be omitted. By Lemma 2.1, we know that $\lim_{t\to\infty} x(t) = 0$. Suppose first, that condition $C.1.3$ is satisfied. Then it follows from here and equation (1.1) that

$$\begin{cases} x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) \le 0, \ t \notin S \\ \Delta x(t_k) + \sum_{i=1}^{m} p_{i,0}x(t_k-\tau_i) \le 0, \ \forall t_k \in S. \end{cases}$$

By Lemma 2.2 (bearing in mind that $f(t,\ u_1,\ \cdots,\ u_m) \equiv p_1(t)\,u_1+\cdots+p_m(t)\,u_m$ and $g(t_k,\ u_1,\ \cdots,\ u_m) \equiv p_{1,0}u_1 + \cdots + p_{m,0}u_m$, we see that equation (1.2) has a finally positive solution. This contradicts our initial hypothesis and therefore completes the proof of Theorem 2.1 when condition $C.1.3$ is fulfilled.

Now, let us assume that condition $C.1.4$ is satisfied. Set

$$\begin{cases} P_i(t) = p_i(t)\,\dfrac{f(t,\ x(t-\tau_1),\ \cdots,\ x(t-\tau_m))}{p_1(t)\,x(t-\tau_1) + \cdots + p_m(t)\,x(t-\tau_m)}; \\ P_i(t_k) = p_{i,0}\dfrac{g(t_k,\ x(t_k-\tau_1),\ \cdots,\ x(t_k-\tau_m))}{p_{1,0}x(t_k-\tau_1) + \cdots + p_{m,0}x(t_k-\tau_m)}. \end{cases}$$

Then by the hypothesis, $P_i(t) \ge 0$, the functions p_i and P_i have the same set of zeros ξ_i with the same multiplicities and

$$\lim_{\substack{(t\to\infty \\ t\neq\xi_i}} \frac{p_i(t)}{P_i(t)} = 1.$$

Observe that $x(t)$ is a non-oscillatory solution of the equation

$$\begin{cases} x'(t) + \sum_{i=1}^{m} P_i(t)x(t-\tau_i) =, \ t \notin S \\ \Delta x(t_k) + \sum_{i=1}^{m} P_{i,0}x(t_k-\tau_i) = 0, \ \forall t_k \in S. \end{cases}$$

Then, it follows, by Lemma 2.3, that equation (1.2) has a non-oscillatory solution. This contradicts the hypothesis and thus completes the proof of Theorem 2.1. □

Theorem 2.2. *Let us assume that there exist an arbitrary positive constant* δ*, the functions* $p_1,\ \cdots,\ p_m \in PC\,[[t_0,\ \infty),\ [0,\ \infty)]$ *and* $p_{1,0},\ \cdots,\ p_{m,0} \in R^+$ *such that either*

$$\begin{cases} 0 < f(t,u_1,\cdots,u_m) \le \sum_{i=1}^{m} p_i(t)u_i;\ 0 < g(t_k,u_1,\cdots,u_m) \le \sum_{i=1}^{m} p_{i,0}u_i \\ \text{and f and g are increasing in } u_1,\cdots,u_m \text{ for } u_1,\cdots,u_m \le \delta \end{cases} \tag{2.12}$$

or

$$\begin{cases} 0 > f(t,u_1,\cdots,u_m) \ge \sum_{i=1}^{m} p_i(t)u_i;\ 0 > g(t_k,u_1,\cdots,u_m) \ge \sum_{i=1}^{m} p_{i,0}u_i \\ \text{and f and g are increasing in } u_1,\cdots,u_m \text{ for } u_1,\cdots,u_m \ge -\delta. \end{cases} \tag{2.13}$$

Assume again, that equation (1.2) *has a non-oscillatory solution. Then equation* (1.1) *also has a non-oscillatory solution.*

Proof. We assume that condition (2.12) is satisfied. The proof when condition (2.13) holds is similar and will be omitted. As the negative solution of equation (1.2) is also a solution, we may assume that equation (1.2) has a finally positive solution $\bar{x}(t)$. Choose $T \ge t_0$ such that $\bar{x}(t-\tau_i) > 0$ for $t \ge T$, $1 \le i \le m$. Then from equation (1.2), we see that $\bar{x}(t)$ is decreasing for $t \ge T$ and so $\bar{x}(t)$ is bounded. Therefore, for M sufficiently large,

$$x(t) = \frac{\bar{x}(t)}{M} \le \delta \text{ for } t \le T_{-1}, \tag{2.14}$$

where T_{-1} is defined by (2.3). Clearly, $x(t)$ is also a positive solution of equation (1.2) for $t \geq T_{-1}$. From equation (1.2) and condition (2.12), it follows that

$$\begin{cases} x^{'}(t) + f(t, x(t-\tau_1), \cdots, x(t-\tau_m)) \leq 0, \ t \geq T, \ t \notin S \\ \Delta x(t_k) + g(t_k, x(t_k - \tau_1), \cdots, x(t_k - \tau_m)) \leq 0, \ \forall t_k \in S. \end{cases} \tag{2.15}$$

Then by Lemma 2.2, equation (1.1) has a finally positive solution and the proof of Theorem 2.2 is complete. □

Theorem 2.3. *Let all the conditions of Theorem 2.2 be fulfilled. Also, let*

$$\tau_i \geq 0, \ p_i \geq 0, \ p_{i,0} \geq 0 \text{ and } p_i + p_{i,0} > 0, \ 1 \leq i \leq m. \tag{2.16}$$

If all solutions of equation (1.1) *are oscillatory, then all the solutions of the linearised equation* (1.2) *are also oscillatory.*

Proof. Let us assume that condition (2.12) is satisfied. The case when condition (2.13) holds is similar and is omitted. Let us assume on the contrary, that equation (1.2) has a finally positive solution $y(t)$. It is clear that

$$\lim_{t \to \infty} y(t) = 0$$

and hence there exists $t_0 > 0$ such that

$$0 < y(t) < \delta \text{ for all } t \geq T_{-1}.$$

Assuming that $y(t)$ is the initial function for $T_{-1} \leq t \leq t_0$, equation (1.1) has a solution $x(t)$ which exists at least in some right neighbourhood of the point t_0.

Notice that so long as $x(t)$ exists and $0 < x(t) < \delta$, we discover that

$$\begin{cases} x^{'}(t) = -f(t, x(t-\tau_1), \cdots, x(t-\tau_m)) \geq -\sum_{i=1}^{m} p_i(t) x(t-\tau_i) \\ \Delta x(t_k) = -g(t_k, x(t_k-\tau_1), \cdots, x(t_k-\tau_m)) \geq -\sum_{i=1}^{m} p_{i,0} x(t_k-\tau_i), \end{cases}$$

and thus, in view of Lemma 2.2

$$y(t) \leq x(t).$$

It follows from equation (1.1) that in as much as $x(t)$ exists and remains positive, it is strictly decreasing. Therefore the inequalities

$$0 < y(t) \leq x(t) < \delta$$

hold for all $t \leq t_0$.

This last statement contradicts the assumption that each solution of equation (1.1) is oscillatory and this completes the proof of Theorem 2.3. □

Observe that Theorem 2.3 is a partial converse of Theorem 2.1. By combining both theorems we obtain the following linearised oscillation result.

Corollary 2.1.

Corollary 2.1. *Assume that conditions $C.1.4$,* (2.12)*,* (2.13) *and* (2.16) *hold. Then every solution of equation* (1.1) *oscillates if and only if every solution of its linearised equation* (1.2) *oscillates.*

3 Summary

The object of investigation in the present paper is the oscillation theory for some general non-linear autonomous impulsive differential equations with several delays in the form of equation (1.1). We developed a linearised oscillation theory which is similar to the so-called linearised stability theory of differential and difference equations.

Roughly speaking, we proved that certain non-linear delay impulsive differential equations have the same oscillatory character as the associated linear equations.

Sufficient conditions for the oscillation and non-oscillation of the said non-linear delay impulsive differential equation (1.2) in terms of the oscillation of the solution linear equation are established and vice versa.

From these results, one can obtain a new linearised oscillation results without the restriction that the $p_{i,0}$'s and p_i's in conditions $C.1.3$ and $C.1.4$ are all positive constants and functions respectively.

Competing Interests

The authors declare that no competing interests exist.

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