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Geometrical Methods and Numerical Computations for Prey-Predator Systems

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Abstract

The goal of this paper is the geometrical and numerical study of the main sizes of the mathematical models of prey-predator interactions which are important in determining long-time dynamics, based on the application of various notions from the theory of dynamical systems to the numerical approximation of initial value problems over long-time intervals. The numerical methods are widely used for the study of complicated temporal behavior of dynamical systems, in order to approximate different types of invariants sets or invariant manifolds and also to extract statistical information on the dynamical behavior in the computation of natural invariant measures or almost invariants sets. The present study is a interplay between dynamical systems geometrical theory and computational calculus of dynamical systems, knowing that the theory provides a framework for interpreting numerical observations and foundations for efficient numerical algorithms.

Keywords: Hamilton-Poisson realization; conservation law; Volterra-Lotka equations; Runge-Kutta method.

1 Introduction

This paper is devoted to studying conservation laws for Volterra-Lotka type systems arising from biology and relationship between this in the geometric framework of Classical Mechanics. The Lotka-Volterra model indeed may be the simplest possible predator-prey model. Nevertheless, it is a useful tool containing the basic properties of the real predator-prey systems, and serves as a robust basis from which it is possible to develop more sophisticated models. The Lotka-Volterra model is very important in population modeling.

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The viewpoint is geometric and we also compute and characterize objects of dynamical significance, in order to understanding the mathematical properties observed in numerical computation for dynamical systems.

We will present two very important examples. First example represent so called *variational dynamical systems*, that is dynamical systems described by a system of ordinary differential equations which can be written as the Euler-Lagrange equations associated to Lagrangian L,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right)-\frac{\partial L}{\partial x^i}=0 \tag{1}$$

This example is the 2D prey-predator Lotka-Volterra system [1-4]. This dynamical systems are included in the presymplectic case because the 2-form ω_L associated to the corresponding Lagrangian is degenerate [1]. Finally, we present different versions of the well-known prey-predator 3D Lotka-Volterra system. This system is not a variational dynamical system. However, we can give more Hamilton-Poisson realizations of this bi-Hamiltonian system like in the 2D case [2,5-7].

2 The 2D Lotka-Volterra System

Let us consider the system of ordinary differential equations [8,9]:

$$\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = -cy + dxy \end{cases}, \; a,b,c,d > 0 \tag{2}$$

This system is called *Lotka-Volterra system* and represents a complex biological system model, in which two species x and y live in a limited area, so that individuals of the species y (predator) feed only individuals of species x (prey) and they feed only resources of the area in which they live. Proportionality factors a and c are respectively increasing and decreasing prey and predator populations. If we assume that the two populations come into interaction, then the factor b is decreasing prey population x caused by this predator population y and the factor d is population growth due to this population x. The evolution system (2) can be written in the form of Euler-Lagrange equations (1), where the Lagrangian L is

$$L=\frac{1}{2}\left(\frac{\ln y}{x}\dot{x}-\frac{\ln x}{y}\dot{y}\right)+c\ln x-a\ln y-dx+by$$

and the corresponding Hamiltonian H is

$$H=\frac{\partial L}{\partial \dot{x}}\dot{x}-\frac{\partial L}{\partial \dot{y}}\dot{y}-L=-c\ln x+a\ln y+dx-by$$

Let us remark that the total energy $E_L = H$ is a *conservation law* for prey-predator system (2) and the Lagrangian L is singular.

According to [2] and [7] the Lotka-Volterra equations (2) has the following *Hamilton-Poisson realization* $\dot{x}^i = J\nabla H$, where $H = c\ln x + a\ln y - dx - by$ is the Hamiltonian and $J=\begin{pmatrix} 0 & xy \\ -xy & 0 \end{pmatrix}$ is the Poisson bracket.

3 The 3D Lotka-Volterra System

In [5] and [7] was discussed the next three-dimensional Lotka-Volterra system which models the evolution of competition between three species:

$$\begin{cases} \dot{x} &= x(cy+z+\lambda) \\ \dot{y} &= y(x+az+\mu) \\ \dot{z} &= z(bz+y+\nu) \end{cases} \tag{3}$$

where $a, b, c \in R, \lambda, \mu, \nu > 0$.

Following [5], if $abc = -1$ and $\nu = \mu b - \lambda ab$, then the 3D Lotka-Volterra system (3) admit two conservation laws $H_1 = ab \ln x - b \ln y + \ln z$ and $H_2 = abx + y - az + \nu \ln y - \mu \ln z$, because (3) is a particular case of a *bi-Hamiltonian system*. The dynamics of (3) has two distinct Hamilton-Poisson realizations $\dot{x}^i = J_1 \nabla H_2$ and $\dot{x}^i = J_2 \nabla H_1$, where

$$J_1 = \begin{pmatrix} 0 & cxy & bcxz \\ -cxy & 0 & -yz \\ -bcxz & yz & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & cxy(az+\mu) & cxz(y+\nu) \\ -cxy(az+\mu) & 0 & xyz \\ -cxz(y+\nu) & -xyz & 0 \end{pmatrix}.$$

From $J_1 \nabla H_1 = 0$ and $J_2 \nabla H_2 = 0$ we have that H_1, H_2 are Casimir functions of J_1, J_2, ([9]).

4 A Numerical Study

Numerical integration is a important, actual and active subject due to the today high computer efficiency (speed and memory), being analysed by extensive theory and a vast range of software, platforms or libraries, [10-13]. Taking into account that even for the simplest 2D Lotka-Volterra system, the analytical solution is useless: root of a polynomial with an integral plus the special function Lambert [14,15], we must resort to numerical methods in order to have information about the trajectories. Thus, constructing a Matlab-based numerical code, we approximate and characterize different types of invariants and also extract information on the dynamical behavior and perform comparisons for both different initial conditions associated to the considered problem and for different values of the parameters. In the first stage we focus on the numerical solving of the initial value problems by appropriate numerical methods, such as Runge-Kutta methods (for the 2D case we use a fourth order Runge-Kutta method [16,17] and for the 3D case we used a fifth order Runge-Kutta method, [18].

The *Runge-Kutta mid-point method* for differential equations solves

$$y' = f(y, t), \quad y(a) = A$$

for $a<t<b$ by setting

$$w_0 = A, \; h = \frac{b-a}{N}, \; t_i = a + ih$$

and iteratively solving

$$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2}, w_i + \frac{h f(t_i, w_i)}{2}\right)$$

for i<N, to estimate the solution over the interval.

The most used type is *Runge-Kutta-Fehlberg (rkf45) method,* since it has a good adaptive numeric procedure for solving the initial value problem $y'(t) = f(t, y(t)), y(t_0) = y_0$. It combines fourth-order and fifth-order Runge-Kutta techniques to monitor error and effect a dynamic step reduction strategy with only two function evaluations more than is used in the fixed-step fourth-order Runge-Kutta method. A naive use of the fourth-order version with step sizes of h and $\frac{h}{2}$ to monitor error would require 4 + 7 = 11 function evaluations at each step [17].

The Runge Kutta method presented above can be easily extended to systems of differential equations. Let us consider the following system:

$$\begin{cases} \dot{y}_1 = f_1(x, y_1, y_2, ..., y_n) \\ \dot{y}_2 = f_2(x, y_1, y_2, ..., y_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{y}_n = f_n(x, y_1, y_2, ..., y_n) \end{cases}$$

We search for the solution satisfying the initial conditions

$$y_i(x_0) = y_{i,0}, i = 1,2,...,n$$

Applying 4th order Runge Kutta method, we get the solution at the step i+1 with the formulas:

$$y_{1,i+1} = y_{1,i} + \Delta y_{1,i}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y_{n,i+1} = y_{n,i} + \Delta y_{n,i}$$

The corrections $\Delta y_{n,i}$ from the above relations are calculating with the relations:

$$\Delta y_{1,i} = \frac{1}{6} \cdot k_1^1 + \frac{1}{3} \cdot k_2^1 + \frac{1}{3} \cdot k_3^1 + \frac{1}{6} \cdot k_4^1;$$
$$\Delta y_{2,i} = \frac{1}{6} \cdot k_1^2 + \frac{1}{3} \cdot k_2^2 + \frac{1}{3} \cdot k_3^2 + \frac{1}{6} \cdot k_4^2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\Delta y_{n,i} = \frac{1}{6} \cdot k_1^n + \frac{1}{3} \cdot k_2^n + \frac{1}{3} \cdot k_3^n + \frac{1}{6} \cdot k_4^n$$

and the coefficients $k_1^1, k_2^1, ..., k_4^1, k_1^2, ... k_4^2, ..., k_1^n, ... k_4^n$ have the following form:

$$k_1^j = h \cdot f_j\left(x_i, y_{1,i}, \ldots, y_{n,i}\right), j = 1,2,\ldots,n;$$

$$k_2^j = h \cdot f_j\left(x_i + \frac{h}{2}, y_{1,i} + \frac{k_1^1}{2}, \ldots, y_{n,i} + \frac{k_1^n}{2}\right), \; j = 1,2,\ldots,n;$$

$$k_3^j = h \cdot f_j\left(x_i + \frac{h}{2}, y_{1,i} + \frac{k_2^1}{2}, \ldots, y_{n,\,i} + \frac{k_2^n}{2}\right), j = 1,2,\ldots,n;$$

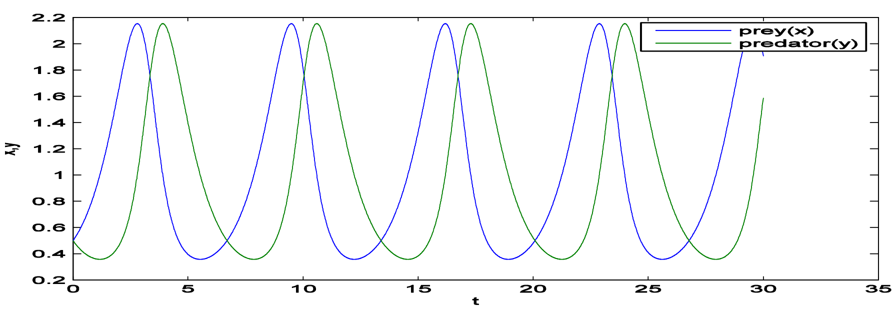
$$k_4^j = h \cdot f_j\left(x_i + h, y_{1,i} + k_3^1, \ldots, y_{n,i} + k_3^n\right), j = 1,2,\ldots,n.$$

Of course a higher order algorithm would produce a better accuracy [17].

We obtain the numerical solution represented by the approximate values of the solution function for a discrete set of data points. Using this approach we perform a numerical analysis of the conservation laws and main sizes.

4.1 2D Lotka-Volterra System

For different values of the parameters and for different initial conditions we represent the profil of the numerical solution, the phase space portrait and the profil of the Lagrangian and Hamiltonian functions.



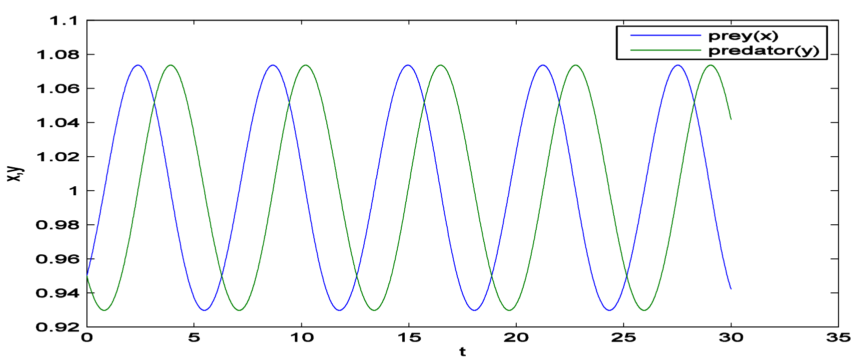


Fig. 1. Profile of the numerical solution (x(t), y(t)), for a = b = c = d = 1 and initial conditions (x0 = 0.5, y0 = 0.5) and (x0 = 0.95, y0 = 0.95)

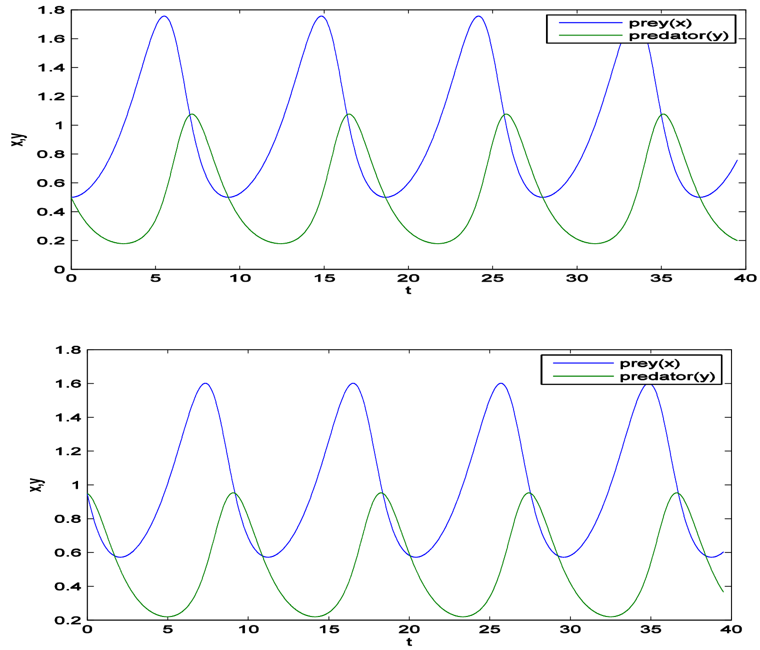


Fig. 2. Profile of the numerical solution (x(t), y(t)), for a = 0.5, b = c = d = 1 and initial conditions (x0 = 0.5, y0 = 0.5), (x0 = 0.95, y0 = 0.95)

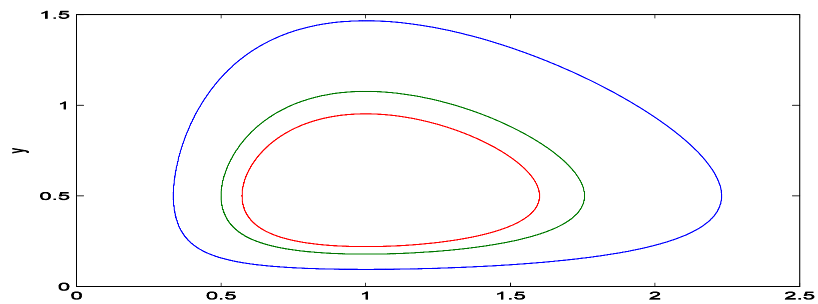


Fig. 3. Phase space portrait for a = 0.5, b =c = d = 1 and initial conditions (x0 =0.35, y0 = 0.35), (x0 = 0.5, y0 = 0.5), (x0 =0.95, y0 = 0.95) listed in order from outermost trajectory to innermost trajectory

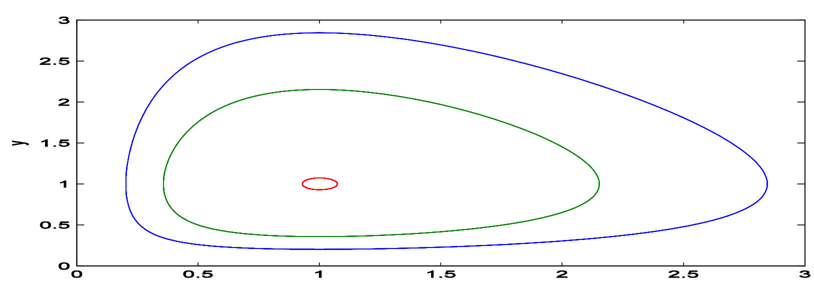


Fig. 4. Phase space portrait for a = 0.5, b =c = d = 1 and initial conditions (x0 =0.35, y0 = 0.35), (x0 = 0.5, y0 = 0.5), (x0 =0.95, y0 = 0.95) listed in order from outermost trajectory to innermost trajectory

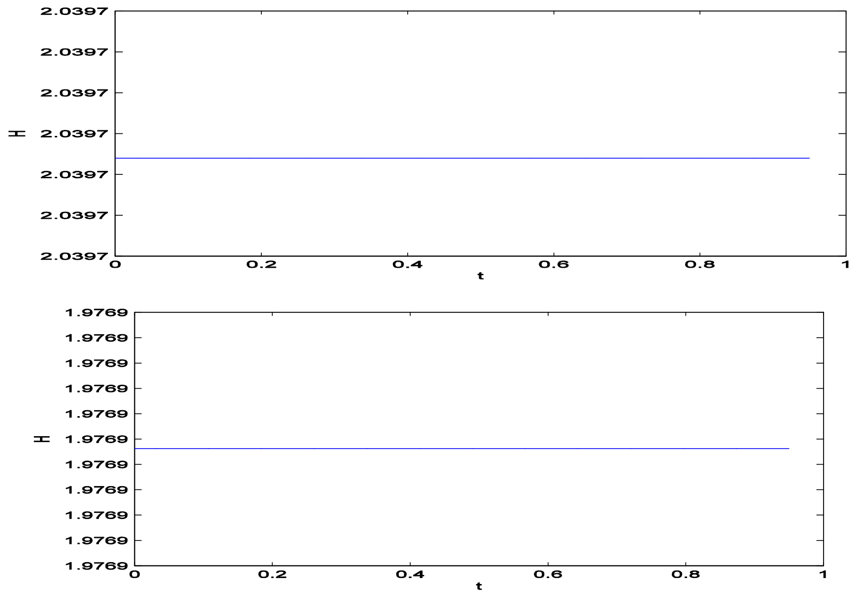


Fig. 5. The profile of the Hamiltonian H as function of t, for a = 0.5, b = c = d = 1 and initial conditions (x0 = 0.5, y0 = 0.5), and (x0 = 0.95, y0 = 0.95) respectively

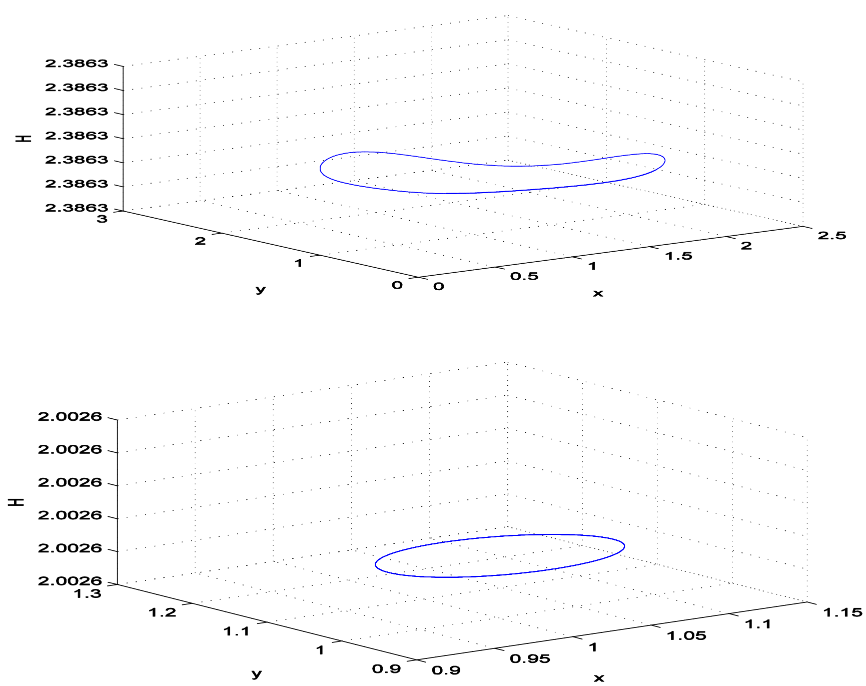


Fig. 6. The profile of H(x, y) for a = 0.5, b =c = d = 1 and initial conditions (x0 = 0.5, y0 =0.5), and (x0 = 0.95, y0 = 0.95) respectively

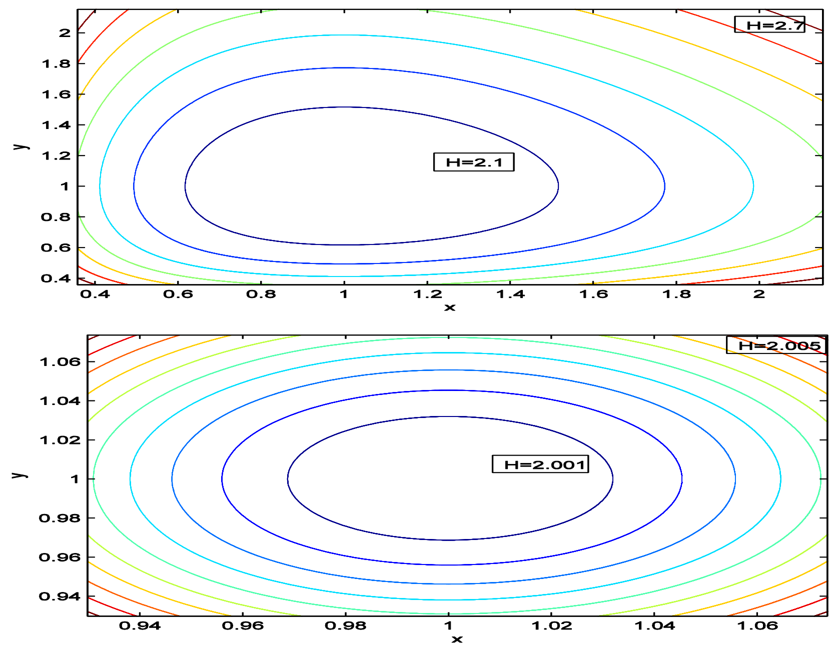


Fig. 7. Some level curves of H(x, y) for a = b = c = d = 1 and initial conditions (x0 = 0.5, y0 = 0.5), and (x0 = 0.95, y0 = 0.95) respectively

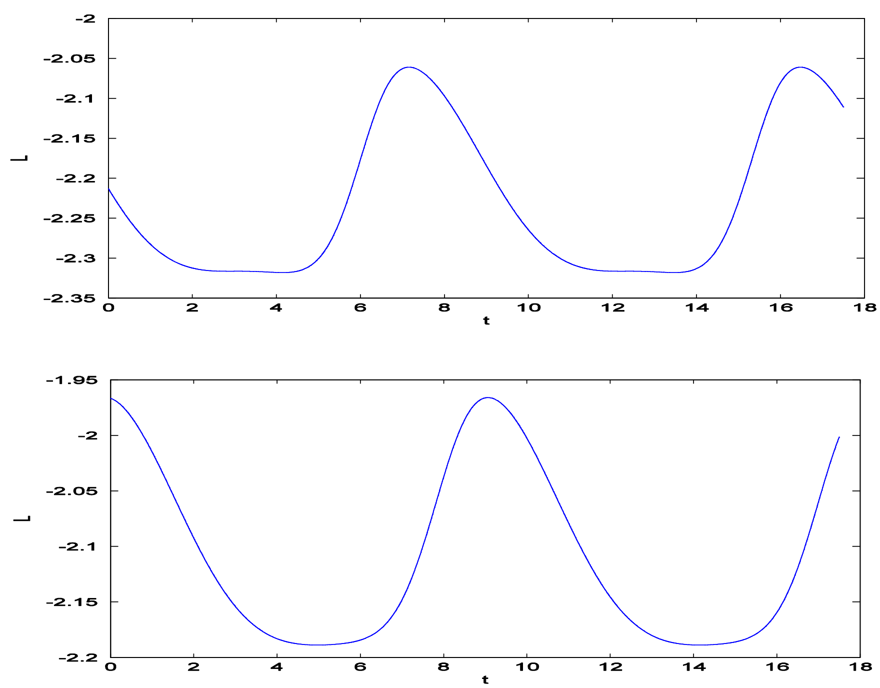


Fig. 8. The profile of the Lagrangian L as function of t for a = 0.5, b = c = d = 1 and initial conditions (x0 = 0.5, y0 = 0.5), and (x0 = 0.95, y0 = 0.95) respectively

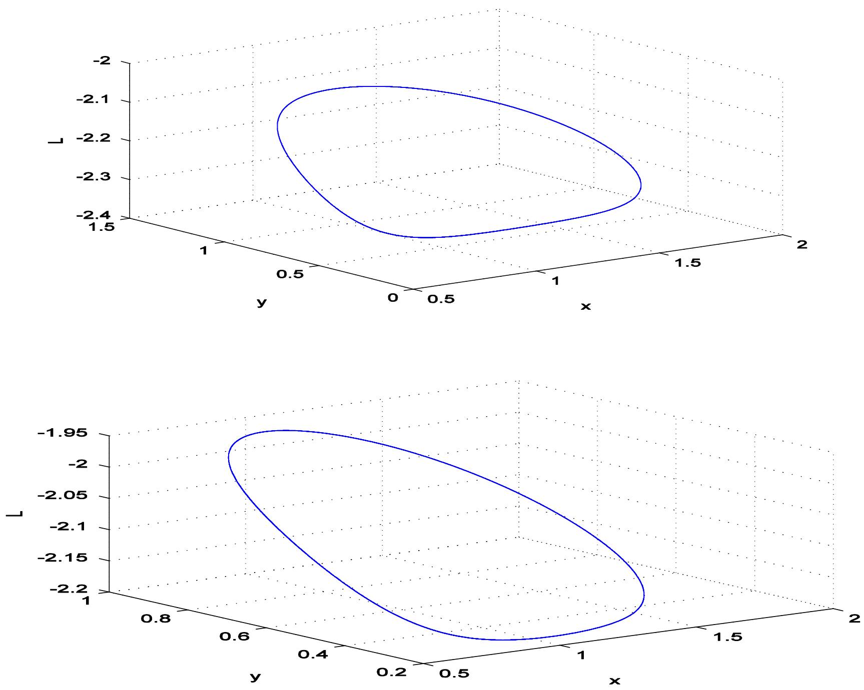


Fig. 9. The profile of L(x, y) for a = 0.5, b =c = d = 1 and initial conditions (x0 = 0.5, y0 = 0.5) and (x0 = 0.95, y0 = 0.95) respectively

4.2 3D Lotka-Volterra System

We consider the case $abc = -1$, $\nu = \mu b - \lambda ab$. We set $a = b = c = -1$ and $\mu = 1$, $\lambda = 0$, $\nu = -1$

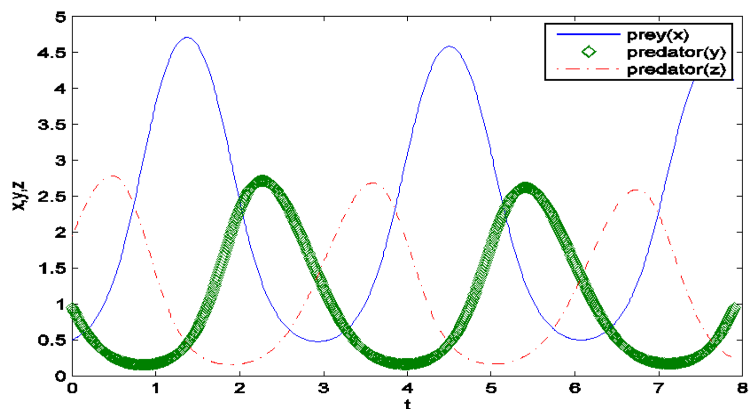


Fig. 10. The profile of the numerical solution (x(t), y(t), z(t)), for the initial conditions (x0 = 0.5, y0 = 1, z0 = 2)

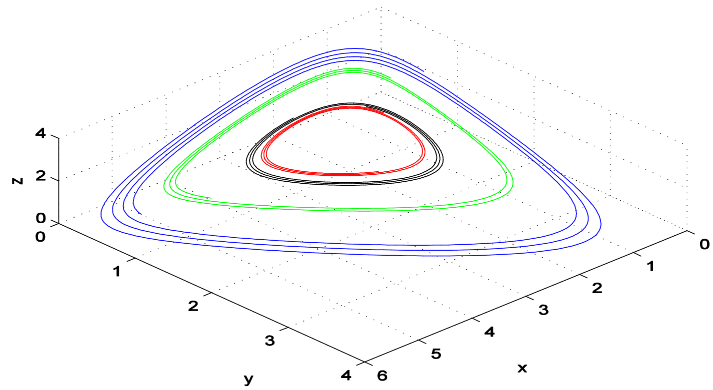


Fig. 11. Phase space portrait for the initial conditions (x0 = 0.5, y0 = 0.95, z0 = 2.95), (x0 = 0.5, y0 = 0.5), z0 = 1.95), (x0 = 1, y0 = 0.75, z0 = 1.25), (x0 = 2.1, y0 = 0.35, z0 = 1.55) listed in order from outermost trajectory to innermost trajectory

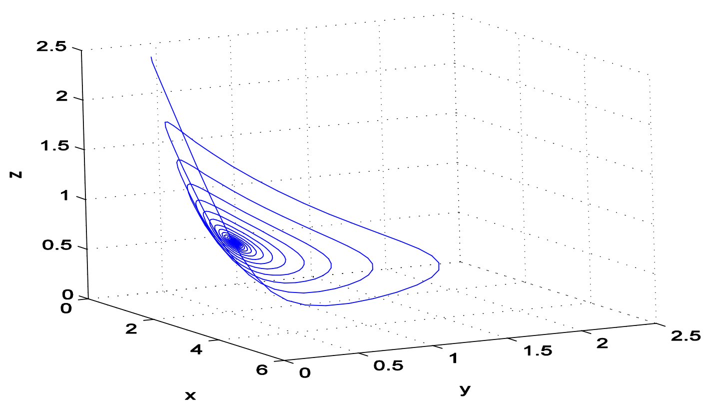


Fig. 12. Phase space for the initial conditions (x0 = 1, y0 = 0.25, z0 = 2.5) The numerical solution presents a graphical profile given by downward spirals

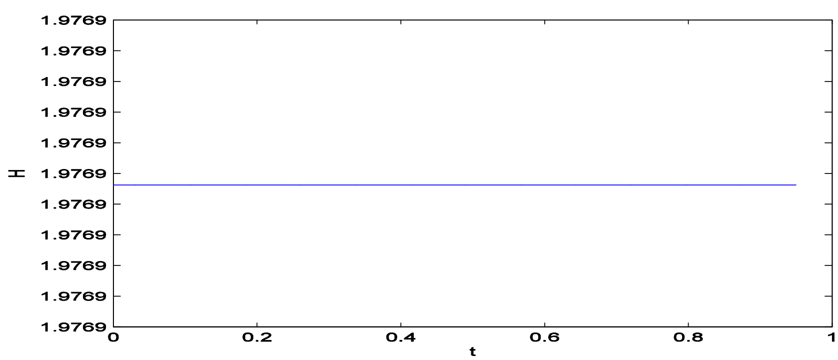


Fig. 13. The profile of the Hamiltonian H1 as a function of t, for the initial conditions (x0 = 2.1, y0 = 0.35, z0 = 1.55)

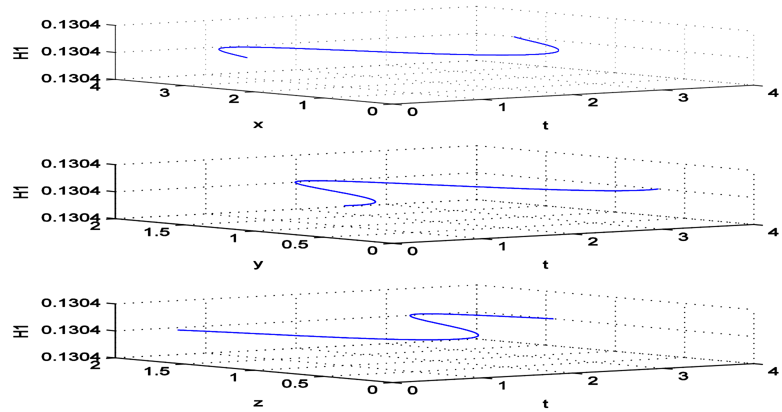


Fig. 14. The profile of the Hamiltonian H1 as a function of (t, x), (t, y) and (t, z), and for the initial conditions (x0 =2.1, y0 = 0.35, z0 = 1.55).

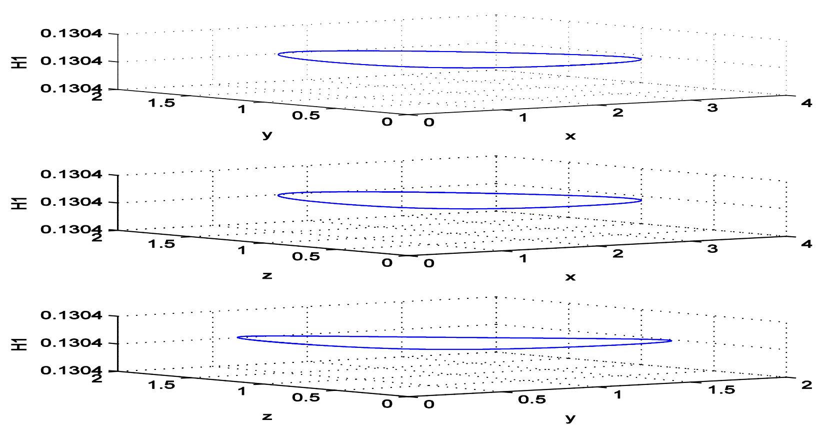


Fig. 15. The profile of the Hamiltonian H1 as a function of (x, y), (x, z) and (y, z), for the initial conditions (x0 = 2.1, y0 = 0.35, z0 = 1.55)

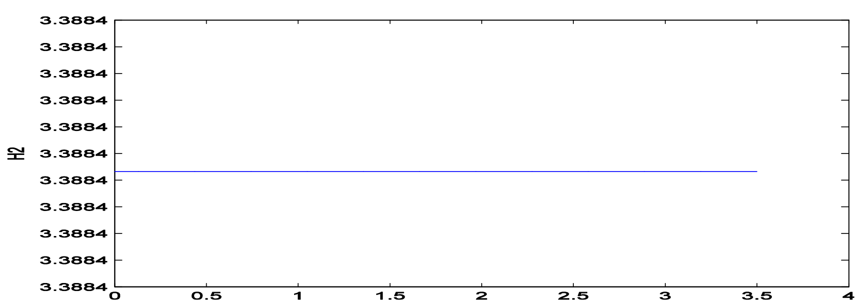


Fig. 16. The profile of the Hamiltonian H2 as a function of t, for the initial conditions (x0 = 2.1, y0 = 0.35, z0 = 1.55)

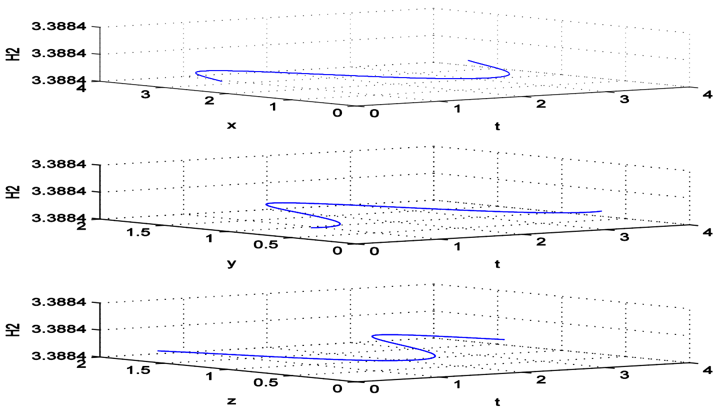


Fig. 17. The profile of the Hamiltonian H2 as a function of (t, x), (t, y) and (t, z) , for the initial conditions (x0 = 2.1, y0 = 0.35, z0 = 1.55).

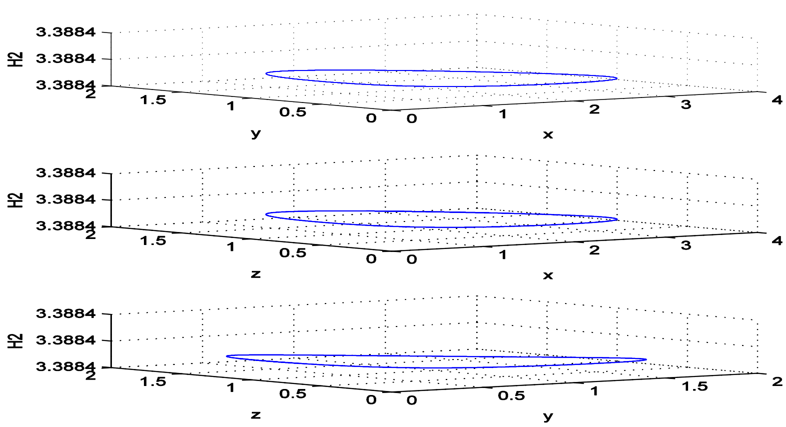


Fig. 18. The profile of the Hamiltonian H2 as a function of (x, y), (x, z) and (y, z), for the initial conditions (x0 = 2.1, y0 =0.35, z0 = 1.55)

5 Results and Discussion

The predator population begins to decline shortly after the prey population starts to decrease. Then after the prey population begins to recover, the predator population also starts to recover [3,4], [19-21]. They share a common period.

The total energy H takes constant values for different values of parameters a, b, c, d and different initial conditions, as we presented in Fig. 5 and Fig. 6. Thus we are in agreement with the property to be a conservation law.

In the 3D case displaying the graph of x, y and z across time t, one observes the periodic behavior of the system. Each predator population also peaks and then begins to decrease shortly after its respective prey population peaks and begins to decrease.

The two Hamiltonians H_1, H_2 associated to the 3D case of Lotka-Volterra system are characterized through our numerical study by constant values, for different initial conditions, as we presented in Figs. 13-18.

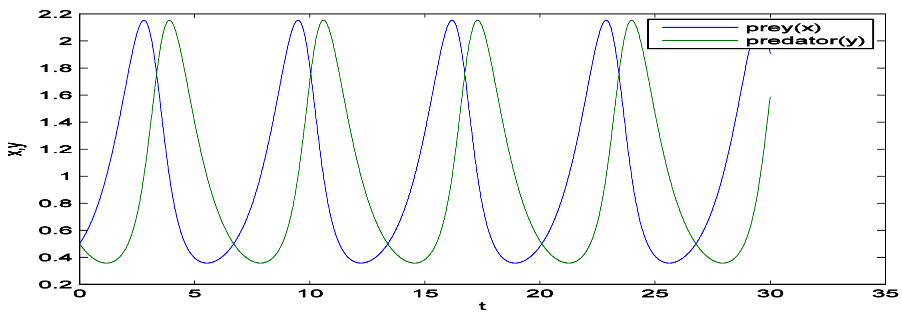


Fig. 19. Profile of the numerical solution for one prey and one predator

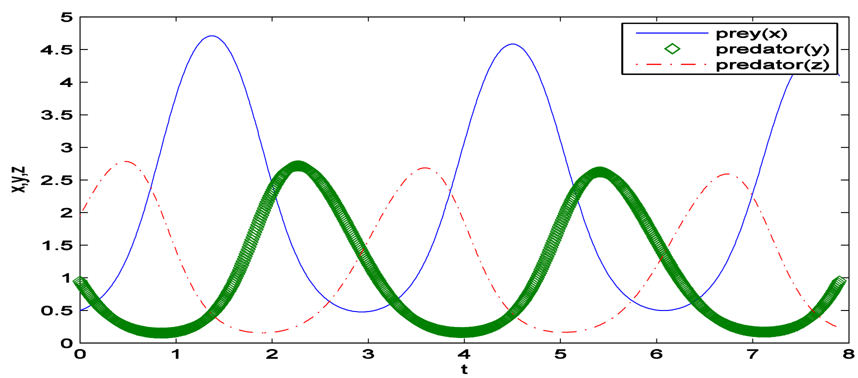


Fig. 20. Profile of the numerical solution for one prey and two predators

6 Conclusion

In this paper we made a computational analysis of the mathematical models of interactions between 2 or 3 species, in order to approximate different types of invariants and main sizes, starting from certain initial value problems associated to these models. Moreover, we obtain the numerical solution and we develop the numerical characterization of the invariants given by the geometrical formalism. Thus, we can make different comparisons between these studied quantities for different values of parameters, for different initial conditions. So, this study is very useful to make certain adjustments for the parameters involved in the equations of the system associated to the model, in order to improve the analyze of the evolution of the populations of prey and predator.

7 Remark

Some part of this manuscript was previously presented and published in the Conference "The VIII-th International Conference of Differential Geometry and Dynamical Systems (DGDS-2014)", September 1-4, 2014, Mangalia, Romania, http://www.mathem.pub.ro/proc/bsgp-22, [21].

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Competing Interests

Authors have declared that no competing interests exist.

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