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On Hermite-Hadamard Inequalities for Differentiable λ-Preinvex Functions via Fractional Integrals

Abdullah Akkurt[1*], M. Esra Yildirim[2] and Hüseyin Yildirim[1]

[1] *Department of Mathematics, Faculty of Science and Arts, University of Kahramanmaraş Sütçü İmam, 46100, Kahramanmaraş, Turkey.*
[2] *Department of Mathematics, Faculty of Science, University of Cumhuriyet, 58140, Sivas, Turkey.*

Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we consider a new class of convex functions which is called λ-preinvex functions. We prove several Hermite–Hadamard type inequalities for differentiable λ-preinvex functions via Fractional Integrals. Some special cases are also discussed.

Keywords: Fractional hermite-hadamard inequalities; preinvex functions; riemann-liouville fractional integrals.

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Corresponding author: E-mail: abdullahmat@gmail.com;

1 Introduction

The convexity property of a given function plays an important role in obtaining integral inequalities. Proving inequalities for convex functions has a long and rich history in mathematics. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hermite-Hadamard inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [1]-[17]) and the references therein.

A significant generalization of convex functions is that of invex functions introduced by Hanson in [7]. Ben-Israel and Mond [2] introduced the concept of preinvex functions, which is a special case of invexity. Note that preinvex functions are nonconvex functions and includes the classical convex functions and its various classes as special cases. Noor [8]-[11] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In recent papers Barani et al. in [1] presented some estimates of the right hand side of a Hermite- Hadamard type inequality in which some preinvex functions are involved. For some recent results related to this nonconvex functions, see the papers ([8]-[11], [12]).

Now, we will give some definitions, lemmas and notations which we use later in this work.

Definition 1.1. ([13]) Let $f \in L_1[a,b]$ .The Riemann-Liouville fractional integral $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\begin{aligned} J_{a+}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt \quad ,a<x \\ J_{b-}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt \quad ,x<b \end{aligned} \tag{1.2}$$

where Γ is the gamma function.

Definition 1.2. ([4]) The incomplete beta function is defined as follows:

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt, \tag{1.3}$$

where $x \in [0,1]$, $a, b > 0$. $B_1(a,b) = B(a,b)$ is so-called beta function.

Definition 1.3. ([16]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class $MT(I)$ if f it is nonnegative, for all $x, y \in I$ and $t \in (0,1)$ satisfies the inequality:

$$f(tx+(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y)\,. \tag{1.4}$$

Lemma 1.1. *([14]) Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) for $a < b$. If $f' \in L[a,b]$, there is the following equality for fractional integrals:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \\ &= \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b)\,dt. \end{aligned} \tag{1.5}$$

Lemma 1.2. *([17]) Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) for $a < b$. If $f'' \in L[a,b]$, there is the following equality for fractional integrals*

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \\ &= \frac{(b-a)^2}{2}\int_0^1 \left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right] f''(ta+(1-t)b)\,dt.\end{aligned} \tag{1.6}$$

Lemma 1.3. *([5]) For $t \in [0,1]$, the following inequalities holds:*

$$(1-t)^m \le 2^{1-m} - t^m \quad for\ m \in [0,1],$$

$$(1-t)^m \ge 2^{1-m} - t^m \quad for\ m \in [1,\infty).$$

Let $\mathbb{R}^n$ be Euclidian space and K is a nonempty closed in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be a continuous functions.

Definition 1.4. ([8]) Let $u \in K$. The set K is said to be invex at u according to η if

$$u + t\eta(v,u) \in K \tag{1.7}$$

for all $u, v \in K$ and $t \in [0,1]$.

Now, we establish new a class of convex functions and then we obtain new Hadamard type inequalities for the new class of convex function.

Definition 1.5. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function. A function f on the set K_η is said to be $\lambda-$preinvex function according to bifunction η and for all $u, v \in I$, $\lambda \in \left(0, \frac{1}{2}\right]$, $t \in (0,1)$, then

$$f(u + t\eta(v,u)) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(u). \tag{1.8}$$

Remark 1.1. In Definition 1.5, if $\lambda = \frac{1}{2}$, and $\eta(v,u) = v - u$. Definition 1.5 reduces to Definition 1.3.

Our goal in this paper is to state and prove the Hermite-Hadamard type inequality for preinvex functions via Riemann-Liouville fractional integrals. In order to achieve our goal, we first give two important lemmas and then by using these identities we prove some integral inequalities.

2 Main Results

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a once differentiable mappings on (a,b) with $a < b$, $\eta(b,a) > 0$. If $f' \in L[a, a+\eta(b,a)]$, then the following equality for fractional integral holds:*

$$\begin{aligned}&\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left[J_{a+}^{\alpha} f(a+\eta(b,a)) + J_{(a+\eta(b,a))-}^{\alpha} f(a)\right] \\ &= \frac{\eta(b,a)}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(a+(1-t)\eta(b,a))\,dt.\end{aligned} \tag{2.1}$$

Proof. Integrating by part and changing the variable of definite integral yield

$$\int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'\left(a+(1-t)\,\eta(b,a)\right) dt$$

$$= \left[(1-t)^\alpha - t^\alpha\right] \frac{f\left(a+(1-t)\,\eta(b,a)\right)}{-\eta(b,a)}\Bigg|_0^1 - \frac{\alpha}{\eta(b,a)} \int_0^1 \left[(1-t)^{\alpha-1} + t^{\alpha-1}\right] f\left(a+(1-t)\,\eta(b,a)\right) dt$$

$$= \frac{f(a) + f\left(a+\eta(b,a)\right)}{\eta(b,a)} - \frac{\alpha}{\eta(b,a)} \left[\frac{1}{(\eta(b,a))^\alpha} \int_a^{a+\eta(b,a)} \left(a+\eta(b,a)-x\right)^{\alpha-1} f(x)\, dx\right.$$

$$\left. + \frac{1}{(\eta(b,a))^\alpha} \int_a^{a+\eta(b,a)} (x-a)^{\alpha-1} f(x)\, dx\right]$$

$$= \frac{f(a) + f\left(a+\eta(b,a)\right)}{\eta(b,a)} - \frac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha+1}} \left[J_{a^+}^\alpha f\left(a+\eta(b,a)\right) + J_{(a+\eta(b,a))^-}^\alpha f(a)\right]. \tag{2.2}$$

By multiplying the both sides of (2.2) by $\frac{\eta(b,a)}{2}$, we have:

$$\frac{f(a) + f\left(a+\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\,(\eta(b,a))^\alpha} \left[J_{a^+}^\alpha f\left(a+\eta(b,a)\right) + J_{(a+\eta(b,a))^-}^\alpha f(a)\right]$$

$$= \frac{\eta(b,a)}{2} \int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'\left(a+(1-t)\,\eta(b,a)\right) dt.$$

Lemma 2.1 is thus proved. □

Remark 2.1. In Lemma 2.1, if $\eta(b,a) = b-a$, Lemma 2.1 reduces to Lemma 1.1.

Theorem 2.2. *Let $I \subseteq \mathbb{R}$ be a open invex set with respect to bifunction $\eta : I \times I \to \mathbb{R}$ where $\eta(b,a) > 0$. Let $f : [0,b] \to \mathbb{R}$ be a differentiable mapping. If $|f'|$ is $\lambda-$preinvex function on I for $\alpha > 0$ and $0 \le a < b$, then:*

$$\left|\frac{f(a) + f\left(a+\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\,(\eta(b,a))^\alpha} \left[J_{a^+}^\alpha f\left(a+\eta(b,a)\right) + J_{(a+\eta(b,a))^-}^\alpha f(a)\right]\right|$$

$$\le \frac{\eta(b,a)}{4} \left\{B_{\frac{1}{2}}\left(\tfrac{3}{2}, \alpha+\tfrac{1}{2}\right) - B_{\frac{1}{2}}\left(\alpha+\tfrac{3}{2}, \tfrac{1}{2}\right) + B_{\frac{1}{2}}\left(\tfrac{1}{2}, \alpha+\tfrac{3}{2}\right) - B_{\frac{1}{2}}\left(\alpha+\tfrac{1}{2}, \tfrac{3}{2}\right)\right\}$$

$$\times \left[\left|f'(a)\right| + \frac{1-\lambda}{\lambda}\left|f'(b)\right|\right].$$

Proof. By using Definition 1.5 and Lemma 2.1, we have:

$$\left|\tfrac{f(a)+f(a+\eta(b,a))}{2} - \tfrac{\Gamma(\alpha+1)}{2(\eta(b,a))^\alpha} \left[J_{a^+}^\alpha f\left(a+\eta(b,a)\right) + J_{(a+\eta(b,a))^-}^\alpha f(a)\right]\right|$$
$$\le \tfrac{\eta(b,a)}{2}\,{}_0^1 \left|(1-t)^\alpha - t^\alpha\right| \left|f'\left(a+(1-t)\,\eta(b,a)\right)\right| dt$$
$$\le \tfrac{\eta(b,a)}{2} \left[{}_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha\right] \left|f'\left(a+(1-t)\,\eta(b,a)\right)\right| dt\right.$$
$$\left. + {}_{\frac{1}{2}}^1 \left[t^\alpha - (1-t)^\alpha\right] \left|f'\left(a+(1-t)\,\eta(b,a)\right)\right| dt\right]$$
$$\le \tfrac{\eta(b,a)}{2} \left[{}_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha\right] \left(\tfrac{\sqrt{t}}{2\sqrt{1-t}} \left|f'(a)\right| + \tfrac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} \left|f'(b)\right|\right) dt\right.$$
$$\left. + {}_{\frac{1}{2}}^1 \left[t^\alpha - (1-t)^\alpha\right] \left(\tfrac{\sqrt{t}}{2\sqrt{1-t}} \left|f'(a)\right| + \tfrac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} \left|f'(b)\right|\right) dt\right]$$

$$\leq \frac{\eta(b,a)}{2}\left[\left|f'(a)\right|_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\frac{1}{2\sqrt{t(1-t)}}dt \right.$$
$$\left. +\frac{(1-\lambda)}{\lambda}\left|f'(b)\right|_0^{\frac{1}{2}}\left[t^{\alpha}-(1-t)^{\alpha}\right]\frac{1}{2\sqrt{t(1-t)}}dt\right]$$
$$\leq \frac{\eta(b,a)}{4}\left\{B_{\frac{1}{2}}\left(\tfrac{3}{2},\alpha+\tfrac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right)+B_{\frac{1}{2}}\left(\tfrac{1}{2},\alpha+\tfrac{3}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right)\right\}$$
$$\times\left[\left|f'(a)\right|+\frac{1-\lambda}{\lambda}\left|f'(b)\right|\right].$$

The proof is done. □

Theorem 2.3. *Let $I=[a,b]\to\mathbb{R}$ be a open invex set with respect to bifunction $\eta: I\times I\to\mathbb{R}$ and $f:[0,b]\to\mathbb{R}$ be a differentiable mapping and $1<q$. If $|f'|^q$ is $\lambda-$preinvex function on I for $0\leq a<b$ and $\eta(b,a)>0$ then:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left[J_{a+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{\eta(b,a)}{2}\left(\frac{\pi}{4}\right)^{\frac{1}{p}}\left(\frac{\pi}{4}\frac{2-2^{1-\alpha p}}{p\alpha+1}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|^q+\tfrac{1-\lambda}{\lambda}\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 1.5, Lemma 2.1 and Hölder's inequality, we have:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left[J_{a+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{\eta(b,a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(a+(1-t)\eta(b,a))\right|dt$$
$$\leq\frac{\eta(b,a)}{2}\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(a+(1-t)\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{\eta(b,a)}{2}\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left(\tfrac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(a)\right|^q+\tfrac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$\leq\frac{\eta(b,a)}{2}\left[\tfrac{\pi}{4}\left|f'(a)\right|^q+\tfrac{\pi}{4}\left(\tfrac{1-\lambda}{\lambda}\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}$$
$$\times\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha p}-t^{\alpha p}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha p}-(1-t)^{\alpha p}\right]dt\right)^{\frac{1}{p}}$$
$$\leq\frac{\eta(b,a)}{2}\left(\tfrac{\pi}{4}\right)^{\frac{1}{p}}\left(\tfrac{\pi}{4}\tfrac{2-2^{1-\alpha p}}{p\alpha+1}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|^q+\tfrac{1-\lambda}{\lambda}\left|f'(b)\right|^q\right)^{\frac{1}{q}}.$$

Here, we $(A_1-A_2)^P\leq A_1^P-A_2^P$ for any $A_1>A_2\geq0$ and $p\geq1$. The proof is done. □

Theorem 2.4. *Let $I=[0,b]\to\mathbb{R}$ be a open invex set with respect to bifunction $\eta: I\times I\to\mathbb{R}$ and $f:[0,b]\to\mathbb{R}$ be a differentiable mapping and $1\leq q$, $f'\in L\left[a+\eta(b,a)\right]$. If $|f'|^q$ is $\lambda-$preinvex* *function on I for $0 \le a < b$ and $\eta(b,a) > 0$ then:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left[J^{\alpha}_{a+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))-}f(a)\right]\right|$$

$$\le 2^{-\frac{1}{q}}\eta(b,a)\left[B_{\frac{1}{2}}\left(\tfrac{3}{2},\alpha+\tfrac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right)+B_{\frac{1}{2}}\left(\tfrac{1}{2},\alpha+\tfrac{3}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right)\right]^{\frac{1}{q}}$$

$$\times\left(\frac{1-2^{-\alpha}}{\alpha+1}\right)^{\frac{q-1}{q}}\left[\frac{|f'(a)|^{q}}{2}+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f'(b)|^{q}}{2}\right]^{\frac{1}{q}}.$$

where $\alpha > 0$.

Proof. By using Definition 1.5, Lemma 2.1 and power mean inequality, we have:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left[J^{\alpha}_{a+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))-}f(a)\right]\right|$$

$$\le \frac{\eta(b,a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(a+(1-t)\eta(b,a))\right|dt$$

$$\le \frac{\eta(b,a)}{2}\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(a+(1-t)\eta(b,a))\right|^{q}dt\right)^{\frac{1}{q}}$$

$$\le \frac{\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(a+(1-t)\eta(b,a))\right|^{q}dt\right)^{\frac{1}{q}}$$

$$\le \frac{\eta(b,a)}{2}\left(\frac{2-2^{1-\alpha}}{\alpha+1}\right)^{\frac{q-1}{q}}\left[\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^{q}+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f'(b)|^{q}\right)dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|^{q}+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f'(b)|^{q}\right)dt\right]^{\frac{1}{q}}$$

$$\le 2^{-\frac{1}{q}}\eta(b,a)\left(\left[B_{\frac{1}{2}}\left(\tfrac{3}{2},\alpha+\tfrac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right)+B_{\frac{1}{2}}\left(\tfrac{1}{2},\alpha+\tfrac{3}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right)\right]\right)^{\frac{1}{q}}$$

$$\times\left(\frac{1-2^{-\alpha}}{\alpha+1}\right)^{\frac{q-1}{q}}\left[\frac{|f'(a)|^{q}}{2}+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f'(b)|^{q}}{2}\right]^{\frac{1}{q}}.$$

The proof is done. □

Lemma 2.5. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mappings on (a,b) with $a<b$, $\eta(b,a)>0$. If $f''\in L[a,a+\eta(b,a)]$, then the following equality for fractional integral holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left[J^{\alpha}_{a+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))-}f(a)\right]\right|$$
$$=\frac{(\eta(b,a))^{2}}{2(\alpha+1)}\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]f''(a+(1-t)\eta(b,a))\,dt. \tag{2.3}$$

Proof. Integrating by part and changing the variable of definite integral yield

$$\begin{aligned}
&\int_0^1 \left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right] f''\left(a+(1-t)\,\eta(b,a)\right)dt \\
&= -\left.\frac{\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right) f'\left(a+(1-t)\,\eta(b,a)\right)}{(\alpha+1)\,\eta(b,a)}\right|_0^1 \\
&+\frac{1}{\eta(b,a)}\int_0^1 \left[(1-t)^{\alpha}-t^{\alpha}\right] f'\left(a+(1-t)\,\eta(b,a)\right)dt \\
&= \frac{1}{\eta(b,a)}\int_0^1 \left[(1-t)^{\alpha}-t^{\alpha}\right] f'\left(a+(1-t)\,\eta(b,a)\right)dt.
\end{aligned} \tag{2.4}$$

Motivated by Lemma 2.1, then:

$$\begin{aligned}
&\frac{1}{\eta(b,a)}\left(\int_0^1 \left[(1-t)^{\alpha}-t^{\alpha}\right] f'\left(a+(1-t)\,\eta(b,a)\right)dt\right) \\
&= \frac{f(a)+f(a+\eta(b,a))}{(\eta(b,a))^2}-\frac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha+2}}\left[J_{a+}^{\alpha} f(a+\eta(b,a))+J_{(a+\eta(b,a))-}^{\alpha} f(a)\right].
\end{aligned} \tag{2.5}$$

By multiplying the both sides of (2.5) by $\frac{(\eta(b,a))^2}{2}$, we have (2.3). The proof is done. □

Remark 2.2. In Lemma 2.5, $\eta(b,a)=b-a$. Lemma 2.5 reduces to Lemma 1.2.

Theorem 2.6. *Let $f:[0,b]\to\mathbb{R}$ be a differentiable mapping. If $|f''|$ is $\lambda-$preinvex function on $[0,b]$ for $0\le a<b$, $\eta(b,a)>0$ and $\alpha>0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left[J_{a+}^{\alpha} f(a+\eta(b,a))+J_{(a+\eta(b,a))-}^{\alpha} f(a)\right]\right| \\
&\le \frac{(\eta(b,a))^2}{4(\alpha+1)}\left\{|f''(a)|\left[\tfrac{\pi}{2}-B\left(\tfrac{3}{2},\alpha+\tfrac{3}{2}\right)-B\left(\alpha+\tfrac{5}{2},\tfrac{1}{2}\right)\right]\right. \\
&\left.+\left(\frac{1-\lambda}{\lambda}\right)|f''(b)|\left[\tfrac{\pi}{2}-B\left(\tfrac{1}{2},\alpha+\tfrac{5}{2}\right)-B\left(\alpha+\tfrac{3}{2},\tfrac{3}{2}\right)\right]\right\}.
\end{aligned}$$

Proof. By using Definition 1.5 and Lemma 2.5, we have:

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left[J_{a+}^{\alpha} f(a+\eta(b,a))+J_{(a+\eta(b,a))-}^{\alpha} f(a)\right]\right| \\
&\le \frac{(\eta(b,a))^2}{2}\int_0^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\left(a+(1-t)\,\eta(b,a)\right)\right|dt \\
&\le \frac{(\eta(b,a))^2}{2(\alpha+1)}\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f''(a)|+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f''(b)|\right)dt
\end{aligned}$$

$$\leq \frac{(\eta(b,a))^2}{2(\alpha+1)}\left\{\frac{|f''(a)|}{2}\left(\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{-1}{2}}dt-\int_0^1 t^{\frac{1}{2}}(1-t)^{\alpha+\frac{1}{2}}dt-\int_0^1 t^{\alpha+\frac{3}{2}}(1-t)^{\frac{-1}{2}}dt\right)\right.$$

$$\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f''(b)|}{2}\left(\int_0^1 t^{\frac{-1}{2}}(1-t)^{\frac{1}{2}}dt-\int_0^1 t^{\frac{-1}{2}}(1-t)^{\alpha+\frac{3}{2}}dt-\int_0^1 t^{\alpha+\frac{1}{2}}(1-t)^{\frac{1}{2}}dt\right)\right\}$$

$$\leq \frac{(\eta(b,a))^2}{4(\alpha+1)}\left\{|f''(a)|\left[\tfrac{\pi}{2}-B\left(\tfrac{3}{2},\alpha+\tfrac{3}{2}\right)-B\left(\alpha+\tfrac{5}{2},\tfrac{1}{2}\right)\right]\right.$$

$$\left.+\left(\frac{1-\lambda}{\lambda}\right)|f''(b)|\left[\tfrac{\pi}{2}-B\left(\tfrac{1}{2},\alpha+\tfrac{5}{2}\right)-B\left(\alpha+\tfrac{3}{2},\tfrac{3}{2}\right)\right]\right\}.$$

The proof is done. □

Theorem 2.7. *Let $f:[0,b]\to\mathbb{R}$ be a differentiable mapping and $1<q$. If $|f''|^q$ is $\lambda-$preinvex function on $[0,b]$ for $\eta(b,a)>0$ and $0\leq a<b$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^\alpha}\left[J_{a^+}^\alpha f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(\eta(b,a))^2}{2(\alpha+1)}\left(1-2^{1-\alpha}\right)\left(\tfrac{\pi}{4}|f''(a)|^q+\tfrac{\pi}{4}\left(\tfrac{1-\lambda}{\lambda}\right)|f''(b)|^q\right)^{\frac{1}{q}}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 1.5, Lemma 2.5, Lemma 1.3 and Hölder's inequality we have:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^\alpha}\left[J_{a^+}^\alpha f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(\eta(b,a))^2}{2}\int_0^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right||f''(a+(1-t)\eta(b,a))|\,dt$$

$$\leq \frac{(\eta(b,a))^2}{2(\alpha+1)}\left(\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(a+(1-t)\eta(b,a))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(\eta(b,a))^2}{2(\alpha+1)}\left(\int_0^1\left[1-2^{-\alpha}\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f''(a)|^q+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f''(b)|^q\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(\eta(b,a))^2}{2(\alpha+1)}\left(1-2^{-\alpha}\right)\left(\tfrac{\pi}{4}|f''(a)|^q+\tfrac{\pi}{4}\left(\tfrac{1-\lambda}{\lambda}\right)|f''(b)|^q\right)^{\frac{1}{q}}.$$

The proof is done. □

Theorem 2.8. *Let $f:[0,b]\to\mathbb{R}$ be a differentiable mapping and $1\leq q$. If $|f''|^q$ is $\lambda-$preinvex function on $[0,b]$ for $0\leq a<b$ and $\eta(b,a)>0$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^\alpha}\left[J_{a^+}^\alpha f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(\eta(b,a))^2}{2(\alpha+1)}\left(1-2^{-\alpha}\right)^{\frac{q-1}{q}}\left(\frac{|f''(a)|^q}{2}\left[B\left(\tfrac{3}{2},\alpha+\tfrac{3}{2}\right)+B\left(\alpha+\tfrac{5}{2},\tfrac{1}{2}\right)-\tfrac{\pi}{2}\right]\right.$$

$$\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f''(b)|^q}{2}\left[B\left(\tfrac{1}{2},\alpha+\tfrac{5}{2}\right)+B\left(\alpha+\tfrac{3}{2},\tfrac{3}{2}\right)-\tfrac{\pi}{2}\right]\right)^{\frac{1}{q}}.$$

where $\alpha > 0$.

Proof. By using Definition 1.5, Lemma 2.5, Lemma 1.3 and power mean's inequality, we have:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{\Gamma(\alpha+1)}{2(\eta(b,a))^{\alpha}}\left[J_{a+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))-}^{\alpha}f(a)\right]\right|$$

$$\leq \frac{(\eta(b,a))^2}{2}\int_0^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''(a+(1-t)\eta(b,a))\right|dt$$

$$\leq \frac{(\eta(b,a))^2}{2(\alpha+1)}\left(\int_0^1\left|1-(1-t)^{\alpha+1}-t^{\alpha+1}\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|1-(1-t)^{\alpha+1}-t^{\alpha+1}\right|\left|f''(a+(1-t)\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(\eta(b,a))^2}{2(\alpha+1)}\left(\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''(a)\right|^q+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\leq \tfrac{(\eta(b,a))^2}{2(\alpha+1)}\left(1-2^{-\alpha}\right)^{1-\frac{1}{q}}$$

$$\times\left(\tfrac{\left|f''(a)\right|^q}{2}\left(\int_0^1 t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt-\int_0^1 t^{\frac{1}{2}}(1-t)^{\alpha+\frac{1}{2}}dt-\int_0^1 t^{\alpha+\frac{3}{2}}(1-t)^{-\frac{1}{2}}dt\right)\right.$$

$$\left.+\left(\tfrac{1-\lambda}{\lambda}\right)\tfrac{\left|f''(b)\right|^q}{2}\left(\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}dt-\int_0^1 t^{-\frac{1}{2}}(1-t)^{\alpha+\frac{3}{2}}dt-\int_0^1 t^{\alpha+\frac{1}{2}}(1-t)^{\frac{1}{2}}dt\right)\right)^{\frac{1}{q}}$$

$$\leq \tfrac{(\eta(b,a))^2}{2(\alpha+1)}\left(1-2^{-\alpha}\right)^{1-\frac{1}{q}}\left(\tfrac{\left|f''(a)\right|^q}{2}\left(\tfrac{\pi}{2}-B\left(\tfrac{3}{2},\alpha+\tfrac{3}{2}\right)-B\left(\alpha+\tfrac{5}{2},\tfrac{1}{2}\right)\right)\right.$$

$$\left.+\left(\tfrac{1-\lambda}{\lambda}\right)\tfrac{\left|f''(b)\right|^q}{2}\left(\tfrac{\pi}{2}-B\left(\tfrac{1}{2},\alpha+\tfrac{5}{2}\right)-B\left(\alpha+\tfrac{3}{2},\tfrac{3}{2}\right)\right)\right)^{\frac{1}{q}}.$$

The proof is done. □

3 Conclusion

In the present paper, we consider a new class of convex functions which is called λ-preinvex functions. We prove several Hermite–Hadamard type inequalities for differentiable λ-preinvex functions via Fractional Integrals. Some special cases are also discussed.

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Competing Interests

Authors have declared that no competing interests exist.

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