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On the Dynamics of an Epidemic Model with Multiple Transmission Ways and Pollution

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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ABSTRACT

An epidemiological model with existence of disease and pollutant has been proposed and analyzed, the formulation of the model is described and the uniqueness and boundness of solution of the system are discussed. The stability analysis of the non-negative equilibrium points is performed. Finally numerical simulation is used to show that increasing the values of external disease transmit ion rate and rates at which the susceptible, infected and recovered individuals decreasing due to the toxicant in proposed model, the population will die out.

Keywords: Epidemic model; equilibrium; stability analysis; toxicant.

1. INTRODUCTION

Infectious disease like**,** influenza, bird flu…etc, and pollutants like oxides of sculpture or oxides of carbon, are the world's leading killer. There are many sources to spread the disease among the population. One of the most ways to spread infectious disease is by contact between the susceptible and infected individual. Moreover many diseases are transmitted in the species not only through contact, but also directly from environment (for example several air-borne diseases such as, influenza, bird f, etc). On the other hand Pollutants may be emitted into the

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environment from different sources (e.g. vehicles, thermal power plant, industries, refineries, etc.) as well as by the incessant use of natural resources without recharging and cleaning them. There are many attempt for describing spread infectious disease among population see [1-4] as well as, in the recent decades, several investigators have proposed and analyzed mathematical models to study the effects of toxicants on biological species [5-16]. In particular, Hallam et al. [17,18] have proposed and analyzed mathematical models to study the effects of toxicants on biological species when these are emitted into the environment from external sources. Hauping and Zhien [9] have proposed a mathematical model to study the effect of a toxicant on natural stable two species communities. In this paper, we proposed and analyzed a mathematical model that describe the spread of infectious disease by two transmission ways under the effect of toxicant on entire population.

2. THE MODEL

Consider an epidemiological system under the following assumptions

1. The population has SIR epidemic disease that divide the population in to three classes namely susceptible $S(t) \geq 0$, infected $I(t) \geq 0$ and recovered $R(t) \geq 0$. Further it is assumed the whole population can reproduce only susceptible logistically with intrinsic growth rate $r > 0$ and charring capacity$K > 0$.
2. The disease is transmitted from infected individuals to susceptible individuals by contact according to non-linear incidence rate of the form $\frac{\lambda SI}{1+I}$.
   Further, it is assumed that the disease is also transmitted to susceptible individuals by an external source with external rate $m > 0$. However the infected individuals may recover and become unsusceptible with recover rate$\alpha > 0$.
3. It is assumed that there are toxicants (pollutants) in the environment which affect negatively on the growth of the whole population (susceptible, infected and recovered). Therefore, if it is assumed that, $W(t)$ is the toxicant concentration in the whole population at time t; $Z(t)$ is the environment concentration of toxicant at time t. Consequently, the dynamics of the epidemic described in above assumptions in a polluted environment can be described by the following set of equations:

$$\frac{dS}{dt} = r(S+I+R)\left(1-\frac{(S+I+R)}{K}\right) - \frac{\lambda IS}{1+I} - mS - \sigma_1 WS$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+I} + mS - (\alpha+\mu_1)I - \sigma_2 WI$$

$$\frac{dR}{dt} = \alpha I - \mu_1 R - \sigma_3 WR \qquad (1)$$

$$\frac{dZ}{dt} = \pi - \tau Z(S+I+R) - \mu_2 Z$$

$$\frac{dW}{dt} = \tau Z(S+I+R) - \mu_3 W$$

Where $S(0), I(0), R(0), Z(0) \text{and } W(0)$ are non negative.

Here, $u_1 > 0$ is natural death rate of infected as well recovered individuals; σ_1, σ_2 and σ_3 are positive rate at which the susceptible, infected and recovered individuals decreasing due to the toxicant; τ is uptake rate of the toxicant by organism ; $\pi > 0$ is the exogenous input rate of toxicant in the environment; μ_2 is the natural depletion of the environmental toxicant and μ_3 is the natural washout rate of the toxicant from the organism. In addition, since the density of population cannot be negative then the state space of the system is

$$R_+^5 = \{(S,I,R,Z,W) \in R^5 : S \geq 0, I \geq 0, R \geq 0, Z \geq 0, W \geq 0\}.$$

Obviously the right side of the system (1) are continuous functions of S,I,R,Z and W and have continuous partial derivatives on the state space R_+^5, therefore these functions are Lipschizian on R_+^5 and then the solution of the system with non negative initial condition exists and is unique. In addition, all the solutions of the system which initiate in the above state space are uniformly bounded as shown in the following theorem.

Theorem (1)

All the solutions of the system (1) that initiate in the state space R_+^5 are uniformly bounded.

Proof. Let $(S(t), I(t), R(t), Z(t), W(t))$ be any solution of the system with the non-negative initial conditions. From the first three equations it is obtained that

$$\frac{d(S+I+R)}{dt} \le r(S+I+R)\left(1-\frac{(S+I+R)}{K}\right)$$

Then by solving the above differential inequality, it is obtained that $\lim_{t\to\infty} Sup(S+I+R) < K$. from the last two equations of the system it is obtained that

$$\frac{d(Z+W)}{dt} = \pi - \mu_2 Z - \mu_3 W$$

Then

$$\frac{d(Z+W)}{dt} < \pi - \mu(Z+W)$$

Here $\mu = \min\{\mu_2, \mu_3\}$. So again by solving the above linear differential inequality, it is obtained that $\lim_{t\to\infty} Sup(Z+W) < \frac{\pi}{\mu}$. Hence all solutions are uniformly bounded and the proof is complete.

3. EQUILIBRIUM POINTS AND LOCAL STABILITY

In this section, the stability analysis of each equilibrium of the system (1) is carried out by using the linearization method with the help of Routh-Huritiz criterion or Lyapunov function. The proposed system may have two equilibrium point the population free equilibrium point $E_1 = \left(0,0,0,\frac{\pi}{\mu_2},0\right)$ and the endemic equilibrium point $E_2 = (\bar{S},\bar{I},\bar{R},\bar{Z},\bar{W})$, where $\bar{S},\bar{I},\ \bar{R},\bar{Z}$ and $\bar{W}$ are positive solution of the following non linear system

$$r(S+I+R)\left(1-\frac{(S+I+R)}{K}\right)-\frac{\lambda IS}{1+I}-mS-\sigma_1 WS=0$$

$$\frac{\lambda IS}{1+I}+mS-(\alpha+\mu_1)I-\sigma_2 WI=0$$

$$\alpha I-\mu_1 R-\sigma_3 WR=0$$

$$\pi-\tau Z(S+I+R)-\mu_2 Z=0$$

$$\tau Z(S+I+R)-\mu_3 W=0$$

The Jacobean matrix for the system at the point E_1 is written as

$$V(E_1)=\begin{pmatrix} r-m & r & r & 0 & 0 \\ m & -\mu_1-\alpha & 0 & 0 & 0 \\ 0 & \alpha & -\mu_1 & 0 & 0 \\ -\tau\frac{\pi}{\mu_2} & -\tau\frac{\pi}{\mu_2} & -\tau\frac{\pi}{\mu_2} & -\mu_2 & 0 \\ \tau\frac{\pi}{\mu_2} & \tau\frac{\pi}{\mu_2} & \tau\frac{\pi}{\mu_2} & 0 & -\mu_3 \end{pmatrix} \quad (2)$$

The eigenvalues of the matrix (2) are the roots of the equation (3)

$$(\lambda+\mu_2)(\lambda+\mu_3)(\lambda^3+D_1\lambda^2+D_2\lambda+D_3)=0 \quad (3)$$

Where

$$D_1 = 2\mu_1+\alpha+m-r$$

$$D_2 = 2\mu_1(m-r)+(m-r)\alpha+\alpha\mu_1+\mu_1^2-rm$$

$$D_3 = \mu_1(\mu_1+\alpha)(m-r)-rm(\mu_1+\alpha)$$

Consequently, the local stability conditions of the first predator free equilibrium point E_1 are established in the following theorem.

Theorem (2):

The population free equilibrium point E_1 of the system (1) is locally asymptotically stable in the $Int.R_+^5$ provided that

$$2\mu_1+m+\alpha > r \quad (4)$$

$$\mu_1(\mu_1+\alpha)(m-r) > rm(\mu_1+\alpha) \quad (5)$$

$$D_1 D_2 > D_3 \quad (6)$$

Proof. It is well known that the equilibrium point E_1 is locally asymptotically stable if and only if all the eigenvalues of $V(E_1)$ have negative real parts. Therefore, from the characteristic equation given by Eq (3). It is clear that the eigenvalue in the $Z-$direction and $W-$direction are $-\mu_2$ and $-\mu_3$, respectively. Which are negative However all the other eigenvalues, which represent the roots of the second part of Eq (3), have negative real parts if and only if $D_1>0$, $D_3>0$ and $D_1D_2-D_3>0$, (using Routh-Hurwitz criterion). Straight forward computation shows that conditions (4-6) guarantee that $D_1>0$, $D_3>0$ and $D_1D_2-D_3>0$. Consequently, all the eigenvalues of $V(E_1)$ have negative real parts under the given conditions and hence the equilibrium point E_1 is locally asymptotically stable.

Now the local stability of the endemic equilibrium point $E_2 = (\bar{S}, \bar{I}, \bar{R}, \bar{Z}, \bar{W})$ of system (1) is discussed below.

The Variation matrix of system at the $E_2 = (\bar{S}, \bar{I}, \bar{R}, \bar{Z}, \bar{W})$ can be written as follows

$V(E_2) = (c_{ij})_{5\times 5}$

Where

$c_{11} = r\left(1 - \frac{2(\bar{S}+\bar{I}+\bar{R})}{K}\right) - \frac{\lambda \bar{I}}{1+\bar{I}} - \sigma_1 \bar{w} - m,$
$c_{12} = r\left(1 - \frac{2(\bar{S}+\bar{I}+\bar{R})}{K}\right) - \frac{\lambda \bar{\bar{S}}}{(1+\bar{I})^2},$
$c_{13} = r\left(1 - \frac{2(\bar{S}+\bar{I}+\bar{R})}{K}\right) c_{14} = c_{23} = c_{24} = c_{31} = c_{34} = c_{45} = 0$, $c_{15} = -\sigma_1 \bar{S}$, $c_{21} = \frac{\lambda \bar{I}}{1+\bar{I}} + m$, $c_{22} = \frac{\lambda \bar{\bar{S}}}{(1+\bar{I})^2} - \mu_1 - \alpha - \sigma_2 \bar{W} c_{25} = -\sigma_2 \bar{I}$, $c_{32} = \alpha$, $c_{33} = -(\mu_1 + \sigma_3 \bar{W})$, $c_{35} = -\sigma_3 \bar{R}$, $c_{41} = c_{42} = c_{43} = -\sigma_4 \bar{Z}$, $c_{44} = -\mu_2 - \sigma_4(\bar{S} + \bar{I} + \bar{R})$, $c_{51} = c_{52} = c_{53} = \sigma_4 \bar{Z}$, $c_{54} = \sigma_4(\bar{S} + \bar{I} + \bar{R})$
and $c_{55} = -\mu_3$

Consequently, the local stability conditions of the endemic equilibrium point E_2 are established in the following theorem.

Theorem (3):

Assume that the positive equilibrium point $E_2 = (\bar{S}, \bar{I}, \bar{R}, \bar{Z}, \bar{W})$ of the system (1) exists. Then it is locally asymptotically stable in the $Int.R_+^4$ provided that

$$r\left(1 - \frac{2(\bar{S} + \bar{I} + \bar{R})}{K}\right) < \frac{\lambda \bar{I}}{1 + \bar{I}} + \bar{w} + m \qquad (7)$$

$$\frac{\lambda \bar{\bar{S}}}{(1 + \bar{I})^2} < \mu_1 + \alpha + \sigma_2 \overline{W} \qquad (8)$$

$$(c_{ij} + c_{ji})^2 < \frac{1}{4} c_{ii} c_{jj} \quad for\ all\ i,j\ and\ i \neq j \qquad (9)$$

$$S \geq \bar{S},\ I \geq \bar{I},\ R \geq \bar{R},\ Z \geq \bar{Z} \text{ and } W \geq \bar{W} \qquad (10)$$

Proof. It is easy to verify that the linearized system of the system (1) can be written as

$$\frac{dX}{dt} = \frac{dU}{dt} = V(E_2)\, U$$

here $X = (S, I, R, Z, W)^T$ and $U = (u_1, u_2, u_3, u_4, u_5)^T$ with $u_1 = S - \bar{S}$, $u_2 = I - \bar{I}$, $u_3 = R - \bar{R}$, $u_4 = Z - \bar{Z}$ and $u_5 = W - \bar{W}$

Now, consider the following function

$$V = \frac{u_1^2}{2} + \frac{u_2^2}{2} + \frac{u_3^2}{2} + \frac{u_4^2}{2} + \frac{u_5^2}{2}$$

Note that $V : R_+^5 \to R$ is a continuously differentiable function that satisfies that

$V(0,0,0,0,0) = 0$ and $V(u_1, u_2, u_3, u_4, u_5) \neq 0$ for all $(u_1, u_2, u_3, u_4) \neq (0,0,0,0,0)$

Hence V is a positive definite function. Now, by differentiating V with respect to time t, it gives

$$\frac{dV}{dt} = u_1 \frac{du_1}{dt} + u_2 \frac{du_2}{dt} + u_3 \frac{du_3}{dt} + u_4 \frac{du_4}{dt} + u_5 \frac{du_5}{dt}$$

Substituting the values of $\frac{du_i}{dt}; i = 1,2,3,4,5$ in the above equation, and after doing some algebraic manipulation; we get that:

$$\frac{dV}{dt} = \sum_{i=1}^{4} \sum_{j=i+1}^{5} \left(\frac{1}{4} c_{ii} u_{ii}^2 + (c_{ij} + c_{ij}) u_i u_j + \frac{1}{4} c_{jj} u_{jj}^2\right)$$

So according to the given conditions (7-10), it is obtained that

$\frac{dV}{dt} < 0,$ Therefore the origin and then $E_2 = (\bar{S}, \bar{I}, \bar{R}, \bar{Z}, \bar{W})$ is a locally asymptotically stable point in the $Int.R_+^5$ and hence the proof is complete.

4. NUMERICAL SIMULATIONS

In this section, the dynamics of the system (1) is investigated numerically to confirm the analytical results system. For the following set of hypothetical, biologically feasible, set of parameters, definitely different set of hypothetical parameters can be chosen. The system is solved numerically starting at different initial points as illustrated in Fig. 1.

$$r = 1, K = 50, \lambda = 0.7\ \mu_1 = 0.6\ , \mu_2 = 0.5, \mu_3 = 0.7$$
$$m = 0.5\ \sigma_1 = \ \sigma_2 = \ \ \sigma_3 = 0.1, \pi = 5, \ \ \alpha = 2,$$
$$\tau = 0.9 \qquad (11)$$

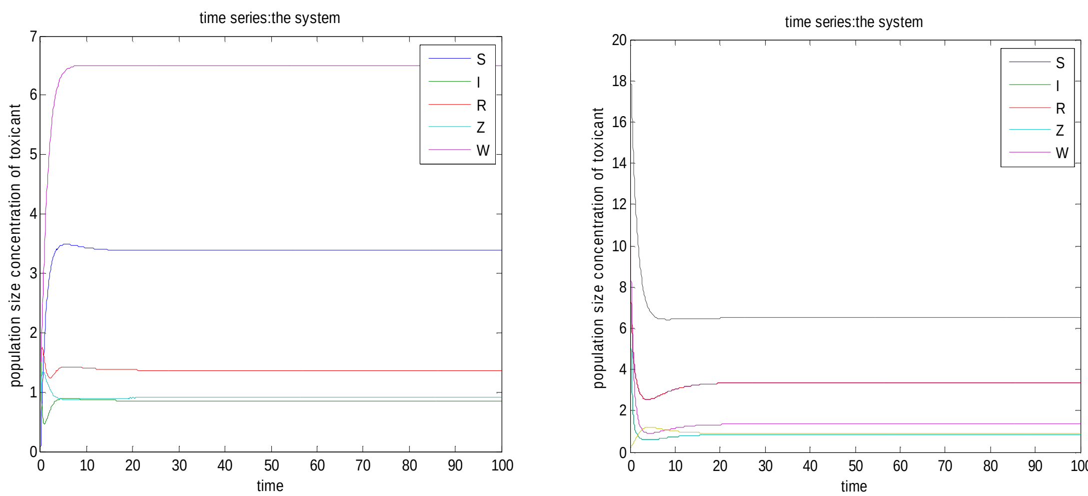

Fig. 1. The solution of system (1) approaches asymptotically the positive equilibrium point $E_2 = (3.3775,\ 0.8556,\ 1.3661, 0.9025, 6.4982)$ For the data given by Eq. (11) starting from two different initial points $(0.1, 1.5, 1.5, 1, 1.5)$ and $(10, 10, 10, 10, 10)$.

Note that it is easy to verify that the data of Fig. 1 Satisfy the stability conditions (7-10), and hence the above figure confirms the analytical results. Further, it is observed that for the above set of data, with $m = 2, \sigma_1 = 0.5,\ \sigma_2 = 0.6,\ \sigma_3 = 0.5,$ the solution of system (1) approaches asymptotically to population free equilibrium point $E_2 = (0,0,0,10,0)$ as shown in Fig. 2.

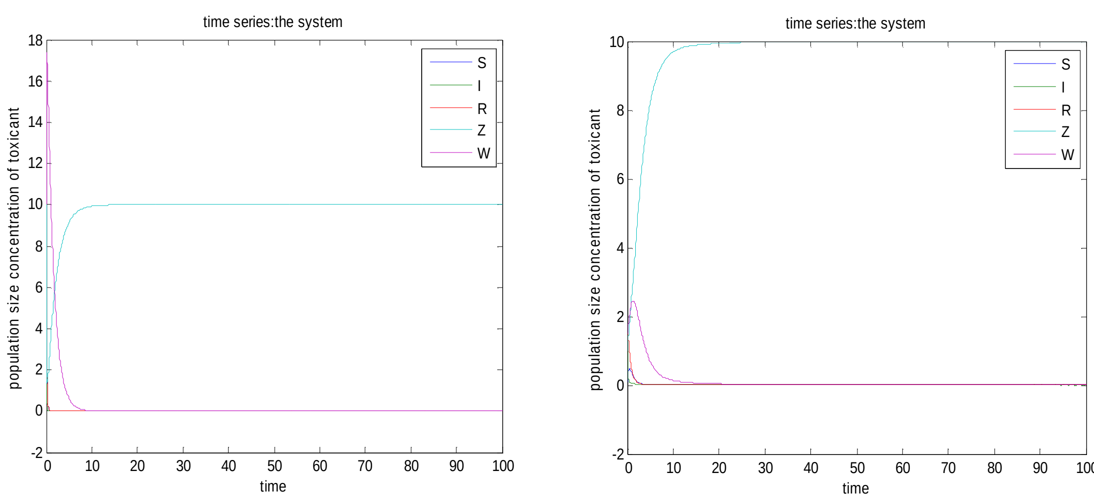

Fig. 2. The solution of the system (1) approaches asymptotically the population free equilibrium point $E_2 = (0, 0, 0, 10, 0)$ for the data given by Eq. (11) with$m = 2, \sigma_1 = 0.5,\ \sigma_2 = 0.6,$ $\sigma_3 = 0.5$

Note that it is easy to verify that the data of Fig. 2 Satisfy the stability conditions (4-6), and hence the above figure confirms the analytical results. "Increasing the values of external transmition rate and rate at which the susceptible, infected and recovered individuals decreasing due to the toxicant, the population will die out".

5. DISCUSSION AND CONCLUSIONS

In this paper, an epidemiological model, with an SIR epidemic disease in the population, is proposed and analyzed. It is assumed that the disease is transmitted through two ways, contact and an external factor, in addition there are toxicants (pollutants) in the environment which affect negatively on the growth of the whole population (susceptible, infected and recovered). The uniqueness and boundness of solution of the system (1) are discussed. The existence of all possible equilibrium points is investigated. The local stability analyses for the proposed system are performed. Moreover, in order to confirm our analytical results and specify which combination of parameters control the dynamical behaviour of system numerical simulations are used for a biologically feasible set of hypothetical parameters. For the set of data given by Eq. (11), and increasing the values of external transmit ion rate and rate at which the susceptible, infected and recovered individuals decreasing due to the toxicant, the population will remain and does not die out.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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