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On the Response to Distributed Loads of Elastic Isotropic Rectangular Plate Moving with Varying Velocities

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Authors' contributions

This work was carried out in collaboration between both authors. Author STO design the study, wrote the protocol and the first draft of the manuscript. Author OKO managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

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Abstract

The dynamic behaviour of a simply supported rectangular plate moving with non-uniform velocities is investigated in this paper. The inertia and gravity effects of the moving load are taken into consideration. In order to solve the governing fourth order partial differential equation, a technique based on two-dimensional finite Fourier sine integral transformations, modification of Struble's asymptotic technique and Fresnel sine and Fresnel cosine identities were used.

The closed form solutions are obtained and numerical analyses in plotted curves are presented. The results show that as the foundation stiffness K and other structural parameters increases, the response amplitude of the simply supported rectangular plate resting on Pasternak foundation decreases. It is also shown that for fixed value of foundation stiffness K, axial force N, shear modulus G and rotatory inertia correction factor R^0, the transverse deflections of the rectangular plate under the action of moving distributed masses are higher than those when only force effects of the moving load is considered. This implies that resonance is reached earlier in moving partially distributed mass problem than in moving distributed force problem.

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1 Introduction

Rectangular plates as the common structural components have been extremely used in various engineering fields such as aerospace, military and marine industries. Such structures are constantly acted upon by moving loads; hence, the problem of analyzing the dynamic response of elastic structures under the action of moving load continues to motivate a variety of investigations [1-6]. The behaviour of plate structures under moving load, in general, is rather complex especially when the inertia effect of the moving loads is taken into consideration [7]. Thus, most of the research works available in literature are those in which the effect has been neglected. This is due, at least in part, to the great amount of computational labour, which is required both to set up and to solve the necessary equations. One important problem that arises when inertia effects of the loads are considered is the singularity which occurs in the inertia terms of the governing differential equation of motion. Generally, the dynamical problems of plate structures under moving load and resting on foundations are complex. The complexity increases if the velocities at which the load moves along the span of the structures are no more constant but a variable function of time. Among the earliest researchers into this subject was Holl [8] who solved the problem of a rectangular plate carrying uniformly moving loads. He concluded that a critical velocity existed for each mode of vibration. Livesly [9] on the other hand, considered the problem of a uniformly travelling load on an infinite plate and showed that there exists a certain critical velocity, beyond which stresses and deflections become infinite. However, in these studies, the plate considered were idealized by one where mass is approximately neglected. Much later Stanisic et al. [10] studied the problem of a simply supported non-Mindlin plate under a Multi-masses moving system and made use of approximation of Dirac delta function and obtained in series form a closed form solution of the dynamical problem. For a plate structure, without an elastic foundation, Willis [11] used the finite element method to study the dynamical response under moving loads. He examined the effects of eccentricity, span length, acceleration and initial velocity of the moving load. Furthermore, the differential quadrature method was shown by Ming-Hung Hsu [12] as an efficient way of obtaining accurate solutions to the problem of rectangular plate resting on an elastic foundation and carrying any number of sprung masses. There was an excellent agreement between the method and known solutions published in literature. Similarly, the problem of non-linear transient dynamic response of clamped rectangular plate on two-parameter foundations was tackled by Civalek et al. [13] using the algorithm of singular convolution. In particular, the problem was discretized in space and time domain using discrete singular convolution (DSC) and harmonic differential quadrature (HDQ) methods respectively. The response to moving concentrated masses of elastic plates on a non-Winkler foundation was later taken up by Gbadeyan and Oni [14]. Very recently, Dongyan et al. [15] use an improved Fourier series method to study free and forced vibration characteristics of moderately thick laminated composite rectangular plates on elastic foundation with uniform and multipoint supports. In all these previous works on the response of plates to moving load, the loads are taken to be moving with constant speed. The more practical cases when the velocities at which these loads move are no longer constants, but vary with time have received little attention in literature. This may be as result of the complex space-time dependencies inherent in such problems. Specifically, even when the inertia effects of the moving load is neglected, analytical solutions involving integral transforms are both intractable and cumbersome [16]. However, such practical problems as acceleration and breaking of automobile on roadways and highway bridges, taking off and landing of air-crafts on runway and breaking and acceleration forces in the calculation of rails and railways bridges in which the motion is not uniform, but a function of time have intensified the need for the study of the behaviour of structures under the action of loads moving with variable velocities. This class of problem was first taken by Lowan [17] who solved the problem of transverse oscillations of beams under the action of moving variable loads. Much later, Kokhmanyuk and Filippov [18] treated the dynamic effects on the transverse motion of a uniform beam of a load moving at variable speed. Recently, the transverse and longitudinal vibration analysis of thin rectangular plate subjected to a variable velocity moving along an arbitrary trajectory using a new finite element method procedure was studied by Ismail [19]. The technique was applied in a simply supported beam-plate structure under a moving load and intensive analysis and simulations were conducted at different dimensionless mass rates. In a more recent development, Ismail [20] studied the transverse and lateral vibration analysis of thin beam under a mass moving with a variable acceleration using a modified finite element method. It is remarked at this juncture, that all the above investigations are very impressive, however only numerical simulation techniques have been employed. Nevertheless, analytical solution is desirable as it often shed more light on some vital information in the vibrating system. In addition to the above, the load in this dynamical system is modeled as distributed load which is a more accurate representation of load models and the foundation is that of Pasternak generally regarded as the preferred alternative.

2 Problem Formulation

The dynamic transverse displacement $V(x, y, t)$ of the mid-surface of a prestressed isotropic rectangular plate resting on bi-parametric Vlasov foundation moving at variable velocity and carrying partially distributed load according to the two-dimensional theory [6] of flexural motions of elastic plate incorporating rotatory inertia correction factor is found by solving

$$\left[D\nabla^2 - \mu R^0 \frac{\partial^2}{\partial t^2}\right]\nabla^2 V(x,y,t) - \left[N_x \frac{\partial^2 V(x,y,t)}{\partial x^2} + N_y \frac{\partial^2 V(x,y,t)}{\partial y^2}\right] + \mu \frac{\partial^2 V(x,y,t)}{\partial t^2} + KV(x,y,t) - G\nabla^2 V(x,y,t) = P(x,y,t) \quad (1)$$

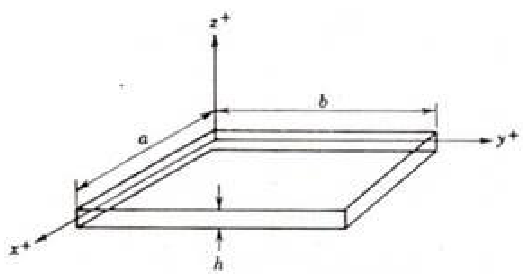

Fig. 1. Isotropic rectangular plate element

where E is the young modulus, ν is the Poisson's ratio $(\nu < 1)$, μ is the mass of the plate per unit length, x is the position coordinate in x-direction, y is the position coordinate in y-direction, t is the time, h is the plate thickness, K is the foundation stiffness, G is the shear modulus and R^0 is the measure of rotatory inertia, ∇^2 is the two-dimensional Laplacian operator, $D = \dfrac{Eh^3}{12(1-\nu)}$ is the bending rigidity of the plate and is constant throughout the plane. The inertia effect of the mass of the partially distributed load on the transverse response of the rectangular plate is considered and the load $P(x, y, t)$ takes the form

$$P(x, y, t) = P_F(x, y, t)\left[1 - \frac{\Delta^*}{g}\left[V(x, y, t)\right]\right] \quad (2)$$

where $P_F(x, y, t)$ is the continuous moving force which travels from point $y = y_0$ on the plate along a straight line parallel to the x-axis with non-uniform velocity c. Thus the $P_F(x, y, t)$ takes the form

$$P_F(x, y, t) = MgH\left[x - x(t)\right]H\left[y - y(t)\right] \quad (3)$$

$$x(t) = x_0 + ct + \tfrac{1}{2}at^2, \qquad y(t) = y_0 \tag{4}$$

g is the acceleration due to gravity and Δ^* is the convective acceleration operator defined as

$$\Delta^* = \frac{\partial^2}{\partial x^2}\left(\frac{dx(t)}{dt}\right)^2 + \frac{\partial^2}{\partial y^2}\left(\frac{dy(t)}{dt}\right)^2 + \frac{\partial^2}{\partial t^2} + 2\frac{\partial^2}{\partial x \partial y}\frac{dx(t)}{dt}\frac{dy(t)}{dt} + 2\frac{\partial^2}{\partial x \partial t}\frac{dx(t)}{dt} + 2\frac{\partial^2}{\partial y \partial t}\frac{dy(t)}{dt} + \frac{\partial}{\partial x}\frac{d^2x(t)}{dt^2} + \frac{\partial}{\partial y}\frac{d^2y(t)}{dt^2} \tag{5}$$

M is the distributed mass and $H(x)$ is the Heaviside function, x_0 and y_0 are the initial positions in the x and y directions respectively. The time t is assumed to be limited to that interval of time within the mass of the plate, that is

$$0 \le ct \le L \tag{6}$$

Thus, in view of equations (2), (3), (4) and (5), equation (1) can be written as

$$\begin{aligned} &D\left[\frac{\partial^4 V(x,y,t)}{\partial x^4} + 2\frac{\partial^4 V(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 V(x,y,t)}{\partial y^4}\right] + \mu\frac{\partial^2 V(x,y,t)}{\partial t^2} - N_x\frac{\partial^2 V(x,y,t)}{\partial x^2} - N_y\frac{\partial^2 V(x,y,t)}{\partial y^2} - \mu R^0\left[\frac{\partial^4 V(x,y,t)}{\partial x^2 \partial t^2} + \frac{\partial^4 V(x,y,t)}{\partial y^2 \partial t^2}\right] \\ &+ KV(x,y,t) - G\left[\frac{\partial^2 V(x,y,t)}{\partial x^2} + \frac{\partial^2 V(x,y,t)}{\partial y^2}\right] + MH\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right]H\left[y - y_0\right]\left[(c+at)^2\frac{\partial^2 V(x,y,t)}{\partial x^2} + \frac{\partial^2 V(x,y,t)}{\partial t^2}\right. \\ &\left. + 2(c+at)\frac{\partial^2 V(x,y,t)}{\partial x \partial t} + a\frac{\partial V(x,y,t)}{\partial x}\right] = MgH\left[x - \left(x_0 + ct + \tfrac{1}{2}at^2\right)\right]H\left[y - y_0\right] \end{aligned} \tag{7}$$

Equation (8) is the fourth order partial differential equation governing the flexural motion of a prestressed isotropic rectangular plate on bi-parametric Vlasov foundation under the action of uniform partially distributed loads moving at non-uniform velocity. The rectangular plate being considered has Equation (8) is the fourth order partial differential equation governing the flexural motion of a prestressed isotropic rectangular plate on bi-parametric Vlasov foundation under the action of uniform partially distributed loads moving at non-uniform spans L_x in the direction of the x – axis and L_y in the direction of the $y -$ axis and is simply supported. Accordingly, the pertinent boundary conditions for $x = 0$ and $x = L_x$ are

$$V(x,y,t) = 0 \qquad \frac{\partial^2 V(x,y,t)}{\partial x^2} = 0 \tag{8}$$

and for $y = 0$, $y = L_y$ are

$$V(x,y,t) = 0 \qquad \frac{\partial^2 V(x,y,t)}{\partial y^2} = 0 \tag{9}$$

The initial conditions are taken to be without any loss of generality.

$$V(x,y,t) = 0 = \frac{\partial V(x,y,t)}{\partial t} \tag{10}$$

3 Solution Procedures

The equation of equation (7) will be obtained by applying

$$\widetilde{V}(j,k,t) = \int_0^{L_x}\int_0^{L_y} V(x,y,t)\sin\frac{j\pi x}{L_x}\sin\frac{k\pi y}{L_y}dxdy \tag{11}$$

with the inverse

$$V(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{4}{L_xL_y}\widetilde{V}(j,k,t)\sin\frac{j\pi x}{L_x}\sin\frac{k\pi y}{L_y} \tag{12}$$

Applying the generalized two-dimensional integral transforms (11), equation (7) can be written as

$$\begin{aligned}&X_AT(0,L_x,L_y)+X_AF_A^0(t)+\widetilde{V}_{tt}(j,k,t)-X_{B1}F_{B1}^0(t)-X_{B2}F_{B2}^0(t)-X_CF_{C1}^0(t)-X_CF_{C2}^0(t)+X_D\widetilde{V}(j,k,t)-X_EF_{B1}^0(t)\\&-X_EF_{B2}^0(t)+F_D^0(t)+F_E^0(t)+F_F^0(t)+F_G^0(t)=\frac{MgL_xL_y}{\mu j\pi k\pi}\left[-\cos k\pi+\cos\frac{k\pi y_0}{L_y}\right]\left[-\cos j\pi+\cos\frac{j\pi}{L_x}(x_0+ct+\tfrac{1}{2}at^2)\right]\end{aligned} \tag{13}$$

where

$$X_A=\frac{D}{\mu},\quad X_{B1}=\frac{N_x}{\mu},\quad X_{B2}=\frac{N_y}{\mu},\quad X_C=R^0,\quad X_D=\frac{K}{\mu},\quad X_E=\frac{G}{\mu}, \tag{14}$$

$$\begin{aligned}T(0,L_x,L_y,t)=&\int_0^{L_y}\left[Sin\frac{j\pi x}{L_x}\frac{\partial^3V(x,y,t)}{\partial x^3}-\frac{j\pi}{L_x}Cos\frac{j\pi x}{L_x}\frac{\partial^2V(x,y,t)}{\partial x^2}+\left(\frac{j\pi}{L_x}\right)^2Sin\frac{j\pi x}{L_x}\frac{\partial V(x,y,t)}{\partial x}+\left(\frac{j\pi}{L_x}\right)^3Cos\frac{j\pi x}{L_x}V(x,y,t)\right]_0^{L_x}Sin\frac{k\pi y}{L_y}dy\\&-\int_0^{L_x}\left[Sin\frac{k\pi y}{L_y}\frac{\partial V(x,y,t)}{\partial y}-\frac{k\pi y}{L_y}Sin\frac{k\pi y}{L_y}V(x,y,t)\right]_0^{L_y}\left(\frac{j\pi}{L_x}\right)^2Sin\frac{k\pi x}{L_x}dx-\int_0^{L_y}\left[Sin\frac{j\pi x}{L_x}\frac{\partial V(x,y,t)}{\partial x}-\frac{j\pi}{L_x}Cos\frac{j\pi x}{L_x}V(x,y,t)\right]_0^{L_x}\left(\frac{k\pi}{L_y}\right)^2Sin\frac{k\pi y}{L_y}dy\\&+\int_0^{L_x}\left[\frac{k\pi}{L_y}\frac{\partial^3V(x,y,t)}{\partial y^3}-\frac{k\pi}{L_y}Sin\frac{k\pi y}{L_y}\frac{\partial^2V(x,y,t)}{\partial y^2}-\left(\frac{ky}{L_y}\right)^2Sin\frac{k\pi y}{L_y}\frac{\partial V(x,y,t)}{\partial y}+\left(\frac{k\pi}{L_y}\right)^3Cos\frac{k\pi y}{L_y}V(x,y,t)\right]_0^{L_y}Sin\frac{j\pi x}{L_x}dx\end{aligned} \tag{15}$$

$$\begin{aligned}F_A^0(t)=&\int_0^{L_x}\int_0^{L_y}V(x,y,t)\left(\frac{j\pi}{L_x}\right)^4Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y}dxdy\\&+2\int_0^{L_x}\int_0^{L_y}V(x,y,t)\left(\frac{j\pi}{L_x}\right)^2\left(\frac{k\pi}{L_y}\right)^2Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y}dxdy+\int_0^{L_x}\int_0^{L_y}V(x,y,t)Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y}dxxdy\end{aligned} \tag{16}$$

$$F_{B1}^0(t)=\int_0^{L_x}\int_0^{L_y}\frac{\partial^2V(x,y,t)}{\partial x^2}Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y}dxdy \tag{17}$$

$$F_{B2}^0(t)=\int_0^{L_x}\int_0^{L_y}\frac{\partial^2V(x,y,t)}{\partial y^2}Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y}dxdy \tag{18}$$

$$F_{C1}^{0}(t) = \int_0^{L_x}\int_0^{L_y} \frac{\partial^4 V(x,y,t)}{\partial x^2 \partial t^2} Sin\,\frac{j\pi x}{L_x}\,Sin\,\frac{k\pi y}{L_y}\,dxdy \tag{19}$$

$$F_{C2}^{0}(t) = \int_0^{L_x}\int_0^{L_y} \frac{\partial^4 V(x,y,t)}{\partial y^2 \partial t^2} Sin\,\frac{j\pi x}{L_x}\,Sin\,\frac{k\pi y}{L_y}\,dxdy \tag{20}$$

$$F_{D}^{0}(t) = \int_0^{L_x}\int_0^{L_y} \frac{M(c+at)^2}{\mu} H\left[x-\left(x_0+ct+\tfrac{1}{2}at^2\right)\right] H\left[y-y_0\right] \frac{\partial^2 V(x,y,t)}{\partial x^2} Sin\,\frac{j\pi x}{L_x}\,Sin\,\frac{k\pi y}{L_y}\,dxdy \tag{21}$$

$$F_{E}^{0}(t) = \int_0^{L_x}\int_0^{L_y} \frac{M}{\mu} H\left[x-\left(x_0+ct+\tfrac{1}{2}at^2\right)\right] H\left[y-y_0\right] \frac{\partial^2 V(x,y,t)}{\partial t^2} Sin\,\frac{j\pi x}{L_x}\,Sin\,\frac{k\pi y}{L_y}\,dxdy \tag{22}$$

$$F_{F}^{0}(t) = \int_0^{L_x}\int_0^{L_y} \frac{2M(c+at)}{\mu} H\left[x-\left(x_0+ct+\tfrac{1}{2}at^2\right)\right] H\left[y-y_0\right] \frac{\partial^2 V(x,y,t)}{\partial x \partial t} Sin\,\frac{j\pi x}{L_x}\,Sin\,\frac{k\pi y}{L_y}\,dxdy \tag{23}$$

$$F_{G}^{0}(t) = \int_0^{L_x}\int_0^{L_y} \frac{aM}{\mu} H\left[x-\left(x_0+ct+\tfrac{1}{2}at^2\right)\right] H\left[y-y_0\right] \frac{\partial V(x,y,t)}{\partial x} Sin\,\frac{j\pi x}{L_x}\,Sin\,\frac{k\pi y}{L_y}\,dxdy \tag{24}$$

$$U_k(y_0) = \frac{L_y}{\lambda_k}\left(-\cos\lambda_k + \cos\frac{\lambda_k y_0}{L_y}\right) \tag{25}$$

And

$$U_j(x_0+ct+\tfrac{1}{2}at^2) = \frac{L_y}{\lambda_j}\left(-\cos\lambda_j + \cos\frac{\lambda_j}{L_x}(x_0+ct+\tfrac{1}{2}at^2)\right) \tag{26}$$

For all pertinent boundary conditions, $T(0, L_x, L_y, t) = 0$ .It is recalled that the equation of the free vibration of a rectangular plate is given by

$$D\left[\frac{\partial^4 V(x,y,t)}{\partial x^4} + 2\frac{\partial^4 V(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 V(x,y,t)}{\partial y^4}\right] + \mu\frac{\partial^2 V(x,y,t)}{\partial t^2} = 0 \tag{27}$$

Substituting

$$V(x,y,t) = Sin\,\frac{j\pi x}{L_x}\,Sin\,\frac{k\pi y}{L_y}\,Cos\,\omega_{j,k}t \tag{28}$$

into the homogeneous part of the equation of the free vibration of the rectangular plate (27), where $\omega_{j,k}$ is the natural circular frequency of a rectangular plate, we obtain

$$D\left[\left(\frac{j\pi}{L_x}\right)^4 Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y} + 2\left(\frac{j\pi}{L_x}\right)^2\left(\frac{k\pi}{L_y}\right)^4 Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y} + \left(\frac{k\pi}{L_y}\right)^4 Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y}\right] - \mu\omega_{j,k}^2 Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y} = 0 \tag{29}$$

It is well known that for a simply supported rectangular plate, $\omega_{j,k}^{2}$ is given by

$$\omega_{j,k}^{2} = D_m\left[\frac{j^4\pi^4}{L_x^4} + 2\frac{j^2k^2\pi^2}{L_x^2L_y^2} + \frac{k^4\pi^4}{L_y^4}\right] \tag{30}$$

Equation (29) implies

$$\begin{aligned}&\int_0^{L_x}\int_0^{L_y} V(x,y,t)\left(\frac{j\pi}{L_x}\right)^4 Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y}dxdy + 2\int_0^{L_x}\int_0^{L_y} V(x,y,t)\left(\frac{j\pi}{L_x}\right)^2\left(\frac{k\pi}{L_y}\right)^2 Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y}dxdy \\ &+\int_0^{L_x}\int_0^{L_y} V(x,y,t)\left(\frac{k\pi}{L_y}\right)^4 Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y}dxdy = \frac{\mu}{D}\omega_{j,k}^2\int_0^{L_x}\int_0^{L_y} V(x,y,t)Sin\frac{j\pi x}{L_x}Sin\frac{k\pi y}{L_y}dxdy\end{aligned} \tag{31}$$

Consequently,

$$F_A^0(t) = \frac{\mu}{D}\omega_{j,k}^2\overline{V}(j,k,t) \tag{32}$$

In order to evaluate the integrals (16) - (20), it is noted that for any arbitrary subscripts $j = p$, $k = q$, equation (12) can be written as

$$V(x,y,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{4}{L_xL_y}\widetilde{V}(p,q,t)Sin\frac{p\pi x}{L_x}Sin\frac{q\pi y}{L_y} \tag{33}$$

It follows that

$$V''(x,y,t) = -\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{4}{L_xL_y}\widetilde{V}(p,q,t)\left(\frac{p\pi}{L_x}\right)^2 Sin\frac{p\pi x}{L_x}Sin\frac{q\pi y}{L_y} \tag{34}$$

Therefore

$$F_{B1}^0(t) = -\frac{p^2\pi^2}{L_x^2}\overline{V}(p,q,t) \tag{35}$$

$$F_{B2}^0(t) = -\frac{q^2\pi^2}{L_y^2}\overline{V}(p,q,t), \tag{36}$$

$$F_{C1}^0(t) = -\frac{p^2\pi^2}{L_x^2}\overline{V}_{tt}(p,q,t) \tag{37}$$

$$F_{C2}^0(t) = -\frac{q^2\pi^2}{L_y^2}\overline{V}_{tt}(p,q,t) \tag{38}$$

In order to evaluate integrals (21), (22), (23) and (24), we make use of the Fourier series representation of the Heaviside function namely,

$$H\left[x-(x_0+ct+\tfrac{1}{2}at^2)\right] = \frac{1}{4}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\left[x-(x_0+ct+\tfrac{1}{2}at^2)\right]}{2n+1} \tag{39}$$

and

$$H\left[y-y_0\right]=\frac{1}{4}+\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)(y-y_0)]}{2m+1} \tag{40}$$

In what follows, we seek to evaluate integral (20) and note that

$$H\left[x-(x_0+ct+\tfrac{1}{2}at^2)\right]H\left[y-y_0\right]=\frac{1}{16}++\frac{1}{4\pi}\sum_{m=0}^{\infty}\frac{Sin(2m+1)\pi(y-y_0)}{2m+1}+\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2m+1}$$
$$+\frac{1}{\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]}{2m+1}\frac{Sin(2m+1)\pi(y-y_0)}{2m+1} \tag{41}$$

In view of (39) the integral (21), can be written as

$$F_F^0(t)=-\frac{M(c+at)^2}{\mu}\left(\frac{p\pi}{L_x}\right)^2\left\{\frac{1}{16}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{4}{L_xL_y}\overline{V}(p,q,t)X(p,j)Y(k,q)+\frac{1}{4\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\frac{4}{L_xL_y}\overline{V}(p,q,t)X(p,j)Y(k,q,m)\right.$$
$$\left.+\frac{1}{4\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty}\frac{4}{L_xL_y}\overline{V}(p,q,t)X(p,j,n)Y(k,q)+\frac{1}{\pi^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{4}{L_xL_y}\overline{V}(p,q,t)X(p,j,n)Y(k,q,m)\right. \tag{42}$$

Making use of similar arguments, it is not difficult to show that

$$F_G^0(t)=\frac{M}{\mu}\left\{\frac{1}{16}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{4}{L_xL_y}\overline{V}_{tt}(p,q,t)X(p,j)Y(k,q)+\frac{1}{4\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\frac{4}{L_xL_y}\overline{V}_{tt}(p,q,t)X(p,j)Y(k,q,m)\right.$$
$$\left.+\frac{1}{4\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty}\frac{4}{L_xL_y}\overline{V}_{tt}(p,q,t)X(p,j,n)Y(k,q)+\frac{1}{\pi^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{4}{L_xL_y}\overline{V}_{tt}(p,q,t)X(p,j,n)Y(k,q,m)\right. \tag{43}$$

$$F_H^0(t)=\frac{2M(c+at)}{\mu}\left\{\frac{1}{16}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{4}{L_xL_y}\overline{V}_t(p,q,t)X_1(p,j)Y(k,q)+\frac{1}{4\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\frac{4}{L_xL_y}\overline{V}_t(p,q,t)X_1(p,j)Y(k,q,m)\right.$$
$$\left.+\frac{1}{4\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty}\frac{4}{L_xL_y}\overline{V}_t(p,q,t)X_1(p,j,n)Y(k,q)+\frac{1}{\pi^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{4}{L_xL_y}\overline{V}_t(p,q,t)X_1(p,j,n)Y(k,q,m)\right. \tag{44}$$

And

$$F_I^0(t)=\frac{aM}{\mu}\left\{\frac{1}{16}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{4}{L_xL_y}\overline{V}(p,q,t)X_1(p,j)Y(k,q)+\frac{1}{4\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\frac{4}{L_xL_y}\overline{V}(p,q,t)X_1(p,j)Y(k,q,m)\right.$$
$$\left.+\frac{1}{4\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty}\frac{4}{L_xL_y}\overline{V}(p,q,t)X_1(p,j,n)Y(k,q)+\frac{1}{\pi^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{4}{L_xL_y}\overline{V}(p,q,t)X_1(p,j,n)Y(k,q,m)\right. \tag{45}$$

Where

$$X(p,j,n)=\int_0^{L_x}\frac{Sin(2n+1)\pi\left[x-(x_0+ct+\frac{1}{2}at^2\right]}{2n+1}Sin\frac{j\pi x}{L_x}Sin\frac{p\pi x}{L_x}dx \tag{46}$$

$$X(p,j)=\int_0^{L_x}Sin\frac{j\pi x}{L_x}Sin\frac{p\pi x}{L_x}dx \tag{47}$$

$$Y(k,q)=\int_0^{L_y}Sin\frac{k\pi y}{L_y}Sin\frac{q\pi y}{L_y}dy \tag{48}$$

$$X_1(p,j) = \int_0^{L_x} Cos\,\frac{p\pi x}{L_x}\,Sin\,\frac{j\pi x}{L_x}\,dx \tag{49}$$

$$X_1(p,j,n) = \int_0^{L_x} \frac{Sin(2n+1)\pi\left[x-(x_0+ct+\frac{1}{2}at^2)\right]}{2n+1} Cos\,\frac{p\pi x}{L_x}\,Sin\,\frac{j\pi x}{L_x}\,dx \tag{50}$$

$$Y(k,q,m) = \int_0^{L_y} \frac{Sin\,(2m+1)\pi\left[y-y_0\right]}{2m+1} Sin\,\frac{k\pi y}{L_y}\,Sin\,\frac{q\pi y}{L_y}\,dy \tag{51}$$

In evaluating the integrals (47)-(49), one takes into account the following orthogonality relations;

$$X(p,j) = \int_0^{L_x} Sin\,\frac{j\pi x}{L_x}\,Sin\,\frac{p\pi x}{L_x}\,dx = \begin{cases} \frac{L_x}{2}, & p = j \\ 0, & p \neq j \end{cases} \tag{52}$$

$$Y(k,q) = \int_0^{Ly} Sin\,\frac{k\pi y}{L_y}\,Sin\,\frac{q\pi y}{L_y}\,dy = \begin{cases} \frac{L_y}{2}, & k = q \\ 0, & k \neq q \end{cases} \tag{53}$$

$$X_1(p,j) = \int_0^{L_x} Cos\,\frac{p\pi x}{L_x}\,Sin\,\frac{j\pi x}{L_x}\,dx = 0, \ \forall\ p, j \tag{54}$$

Noting that

$$\frac{Sin\,(2n+1)\pi\left[x-(x_0+ct+\frac{1}{2}at^2)\right]}{2n+1} = \frac{Sin\,(2n+1)\pi x Cos\,(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1} - \frac{Cos\,(2n+1)\pi x Sin\,(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1} \tag{55}$$

Such that

$$\int_0^{L_x} \frac{Sin\,(2n+1)\pi\left[x-(x_0+ct+\frac{1}{2}at^2)\right]}{2n+1} Sin\,\frac{j\pi x}{L_x}\,Sin\,\frac{p\pi x}{L_x}\,dx = \frac{(2n+1)L_x^2}{2\pi}\left[Z_1(p,j)\cdot\frac{Cos\,(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1} - Z_2(p,j)\frac{Sin\,(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right] \tag{56}$$

Where

$$Z_1(p,j) = \left[\frac{(-1)^{p+j}\cos(2n+1)\pi L_x - 1}{\left[(2n+1)L_x\right]^2 - (p+j)^2} - \frac{(-1)^{p-j}\cos(2n+1)\pi L_x - 1}{\left[(2n+1)L_x\right]^2 - (p-j)^2}\right] \tag{57}$$

$$Z_2(p,j) = \left[\frac{(-1)^{p-j}\sin(2n+1)\pi L_x}{\left[(2n+1)L_x\right]^2 - (p-j)^2} - \frac{(-1)^{p+j}\cos(2n+1)\pi L_x}{\left[(2n+1)L_x\right]^2 - (p+j)^2}\right] \tag{58}$$

and

$$\int_0^{L_x} \frac{\sin(2n+1)\pi\left[x-(x_0+ct+\frac{1}{2}at^2)\right]}{2n+1}\cos\frac{p\pi x}{L_x}\sin\frac{j\pi x}{L_x}\,dx = \frac{(2n+1)L_x}{2\pi}\left[Z_1^*(p,j)\cdot\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1} - Z_2^*(p,j)\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right] \tag{59}$$

where

$$Z_1^*(p,j)=\left[\frac{(p+j)\left\{(-1)^{p+j}\cos(2n+1)\pi L_x-1\right\}}{\left[(2n+1)L_x\right]^2-(p+j)^2}-\frac{(p-j)\left\{(-1)^{p+j}\cos(2n+1)\pi L_x-1\right\}}{\left[(2n+1)L_x\right]^2-(p-j)^2}\right] \tag{60}$$

$$Z_2^*(p,j)=\left[\frac{(p+j)(-1)^{p+j}\sin(2n+1)\pi L_x}{\left[(2n+1)L_x\right]^2-(p+j)^2}-\frac{(p-j)(-1)^{p-j}\sin(2n+1)\pi L_x}{\left[(2n+1)L_x\right]^2-(p-j)^2}\right] \tag{61}$$

$$\int_0^{L_x}\frac{\sin(2m+1)\pi(y-y_0)}{2m+1}\sin\frac{q\pi x}{L_y}\sin\frac{k\pi x}{L_y}dx=\frac{(2m+1)L_y}{2\pi}\cdot \left[Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}-Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\right] \tag{62}$$

$$Z_1(k,q)=\left[\frac{(-1)^{k+q}\cos(2m+1)\pi L_x-1}{\left[(2m+1)L_x\right]^2-(k+q)^2}-\frac{(-1)^{k-q}\cos(2m+1)\pi L_x-1}{\left[(2m+1)L_x\right]^2-(k-q)^2}\right] \tag{63}$$

$$Z_2(k,q)=\left[\frac{(-1)^{k-q}\sin(2m+1)\pi L_x}{\left[(2m+1)L_x\right]^2-(k-q)^2}-\frac{(-1)^{k+q}\cos(2m+1)\pi L_x}{\left[(2m+1)L_x\right]^2-(k+q)^2}\right] \tag{64}$$

Substituting the above results into equation (13), after some rearrangement, one obtains

$$\begin{aligned}
&\widetilde{V}_{tt}(j,k,t)+\left(\omega_{j,k}^2+\frac{K}{\mu}\right)\widetilde{V}(j,k,t)+\left(\frac{N_x j^2\pi^2}{\mu L_x^2}-\frac{N_y k^2\pi^2}{\mu L_y^2}\right)\left(\frac{j^2\pi^2}{L_x^2}-\frac{k^2\pi^2}{L_y^2}\right)\widetilde{V}(j,k,t)+R^0\left(\frac{j^2\pi^2}{L_x^2}-\frac{k^2\pi^2}{L_y^2}\right)^2\widetilde{V}_{tt}(j,k,t)\\
&+\frac{G}{\mu}\left(\frac{j^2\pi^2}{L_x^2}-\frac{k^2\pi^2}{L_y^2}\right)^2\widetilde{V}(j,k,t)+\Gamma_1 L_x L_y\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\left\{(c+at)^2\left[-\frac{p^2\pi^2}{L_x^2}+\frac{\pi}{2L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}p^2q^2\left(Z_2(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\right.\right.\right.\\
&\left.-Z_1(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\right)-\frac{\pi}{L_x^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty}p^2\left(Z_1(p,j)\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}-Z_2(p,j)\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right)\\
&+\frac{4\pi^2}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p^2q^2(2m+1)(2n+1)Z_1(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\\
&-\frac{4\pi^2}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p^2q^2(2m+1)(2n+1)Z_2(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\\
&-\frac{4\pi^2}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p^2q^2(2m+1)(2n+1)Z_1(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\\
&\left.+\frac{4\pi^2}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p^2q^2(2m+1)(2n+1)Z_2(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right]\\
&-a\left[\frac{p}{L_x}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty}\left(Z_1^*(p,j)\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}-Z_2^*(p,j)\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right)\right.\\
&-\frac{4\pi}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2(2m+1)(2n+1)Z_1^*(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}
\end{aligned}$$

$$+\frac{4\pi}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2(2m+1)(2n+1)Z_2^*(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}$$
$$+\frac{4\pi}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2(2m+1)(2n+1)Z_1^*(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}$$
$$\left.-\frac{4\pi}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2(2m+1)(2n+1)Z_2^*(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right]\widetilde{V}(p,q,t)$$

$$+(c+a)\left[\frac{pL_y}{8L_x}+\frac{p}{L_x}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty}\left(Z_1^*(p,j)\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}-Z_2^*(p,j)\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right)\right.$$
$$+\frac{4\pi}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2Z_1^*(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}$$
$$-\frac{4\pi}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2Z_2^*(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}$$
$$-\frac{4\pi}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2Z_1^*(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}$$
$$\left.+\frac{4\pi}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2Z_2^*(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right]\widetilde{V}_t(p,q,t)$$

$$+\left[\frac{L_xL_y}{16}-\frac{\pi}{2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}q^2\left(Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}-Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\right)\right.$$
$$-\frac{L_x}{2\pi}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{n=0}^{\infty}\left(Z_1(p,j)\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}-Z_2(p,j)\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\right)$$
$$-\frac{2\pi^2}{L_x^2L_y}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2(2m+1)(2n+1)Z_1^*(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}$$
$$+\frac{2\pi^2}{L_x^2L_y}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2(2m+1)(2n+1)Z_2^*(p,j)Z_1(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}$$
$$-\frac{2\pi^2}{L_x^2L_y}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2(2m+1)(2n+1)Z_1^*(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}$$
$$+\frac{4\pi}{L_x^2L_y^2}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}pq^2(2m+1)(2n+1)Z_2^*(p,j)Z_2(k,q)\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}\cdot$$
$$\left.\frac{\sin(2m+1)\pi y_0}{2m+1}\right]\widetilde{V}_{tt}(p,q,t)=\frac{MgL_xL_y}{\mu j\pi k\pi}\left[-\cos k\pi+\cos\frac{k\pi y_0}{L_y}\right]\left[-\cos j\pi+\cos\frac{j\pi}{L_x}(x_0+ct+\tfrac{1}{2}at^2)\right] \quad (65)$$

Where

$$\Gamma_1=\frac{M}{\mu L_xL_y} \quad (66)$$

Equation (65) is now the fundamental equation of our problem when the isotropic rectangular plate has simple supports at all its edges. Next, we shall consider two cases of the equation.

3.1 Simply supported rectangular plate traversed by moving distributed force

The moving distributed force model of the simply supported rectangular plate is obtained by setting $\Gamma_1 = 0$ in equation (65). In this case, equation (65) reduces to

$$\tilde{V}_{tt}(j,k,t)+\left(\omega_{j,k}^2+\frac{K}{\mu}\right)\tilde{V}(j,k,t)+\left(\frac{N_x j^2\pi^2}{\mu L_x^2}-\frac{N_y k^2\pi^2}{\mu L_y^2}\right)\left(\frac{j^2\pi^2}{L_x^2}-\frac{k^2\pi^2}{L_y^2}\right)\tilde{V}(j,k,t)+R^0\left(\frac{j^2\pi^2}{L_x^2}-\frac{k^2\pi^2}{L_y^2}\right)^2\tilde{V}_{tt}(j,k,t)$$
$$+\frac{G}{\mu}\left(\frac{j^2\pi^2}{L_x^2}-\frac{k^2\pi^2}{L_y^2}\right)^2\tilde{V}(j,k,t)=\frac{MgL_xL_y}{\mu j\pi k\pi}\left[-\cos k\pi+\cos\frac{k\pi y_0}{L_y}\right]\left[-\cos j\pi+\cos\frac{j\pi}{L_x}(x_0+ct+\tfrac{1}{2}at^2)\right] \tag{67}$$

This is an approximate model which assumes the inertia effect of the moving distributed mass as negligible. Thus equation (67) after some rearrangement can be written as

$$\tilde{V}_{tt}(j,k,t)+\delta_{SMF}^2\tilde{V}(j,k,t)=P_{MF}^0\left[-(-1)^J+\cos\frac{j\pi}{L_x}(x_0+ct+\tfrac{1}{2}at^2)\right] \tag{68}$$

where

$$\delta_{SMF}^2=\frac{\omega_{j,k}^2+\frac{K}{\mu}+\frac{L_xL_y}{4\mu}\left(N_x\frac{j^2\pi^2}{L_x^2}+N_y\frac{k^2\pi^2}{L_y^2}\right)+\frac{G}{\mu}\frac{L_xL_y}{4}\left(\frac{j^2\pi^2}{L_x^2}+\frac{k^2\pi^2}{L_y^2}\right)}{1+R_0\frac{L_xL_y}{4}\left(\frac{j^2\pi^2}{L_x^2}+\frac{k^2\pi^2}{L_y^2}\right)} \tag{69}$$

and

$$P_{MF}^0=\frac{\frac{MgL_xL_y}{\mu j\pi k\pi}\left(-\cos k\pi+\cos\frac{k\pi y_0}{L_y}\right)}{1+R_0\frac{L_xL_y}{4}\left(\frac{j^2\pi^2}{L_x^2}+\frac{k^2\pi^2}{L_y^2}\right)} \tag{70}$$

Solving equation (68) using variation of parameters method in conjunction with Fresnel sine and cosine identities and the initial conditions (10), one obtains

$$\begin{aligned}
V(x,y,t)=\frac{4}{L_xL_y}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{P_{MF}^0\sqrt{\pi}}{2\delta_{SMF}\sqrt{2a}}\Bigg\{&\sin\delta_{SMF}t\Bigg(\sin\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)+\cos\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)\\
&+\sin\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)+\cos\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)-\sin\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2}{\sqrt{2\pi a}}\right)\\
&-\cos\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2}{\sqrt{2\pi a}}\right)-\sin\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1}{\sqrt{2\pi a}}\right)-\cos\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1}{\sqrt{2\pi a}}\right)\\
&-\frac{1}{2\delta_{SMF}}\left[\sin(j\pi+\delta_{SMF}t)-\sin(j\pi-\delta_{SMF}t)\right]\Bigg)-\cos\delta_{SMF}t\Bigg(-\sin\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)\\
&+\cos\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)+\sin\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)-\cos\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)\\
&+\sin\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1}{\sqrt{2\pi a}}\right)-\cos\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1}{\sqrt{2\pi a}}\right)-\sin\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2}{\sqrt{2\pi a}}\right)\\
&+\cos\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2}{\sqrt{2\pi a}}\right)+\frac{1}{\delta_{SMF}}\cos j\pi+\frac{1}{2\delta_{SMF}}\left[\cos(j\pi-\delta_{SMF}t)\right.\\
&\left.-\cos(j\pi+\delta_{SMF}t)\right]\Bigg)\Bigg\}\sin\frac{j\pi x}{L_x}\sin\frac{k\pi y}{L_y}
\end{aligned} \tag{71}$$

Equation (71) is the transverse displacement response to travelling distributed forces moving at variable velocities of a simply supported isotropic rectangular plate with rotatory inertia correction factor resting on a non-Winkler type elastic foundation.

3.2 Simply supported rectangular plate traversed by moving distributed mass

In this section, the mass of the moving load is commensurable with that of the structure and the inertia effect of the moving mass is not considered negligible. Thus, $\Gamma_1 \neq 0$ and the solution to the entire equation (65) when no term of the coupled differential equation is neglected, is required. This is termed the ***moving mass problem***. Thus, in view of the homogeneous part of equation (67), equation (65) can now be written as

$$\widetilde{V}_{tt}(j,k,t)+\frac{\Gamma_1 H_2^*(m,n,t)}{1+\Gamma_1 H_1^*(m,n,t)}\widetilde{V}_t(j,k,t)+\frac{\left(\delta_{SMF}^2+\Gamma_1 H_3^*(m,n,t)\right)}{1+\Gamma_1 H_1^*(m,n,t)}\widetilde{V}(j,k,t)+\frac{\Gamma_1 H_2^*(m,n,t)}{1+\Gamma_1 H_1^*(m,n,t)}\Big\{(c+at)^2\cdot$$

$$\Bigg(\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)(2n+1)p^2q^2\big(Z_1(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}$$

$$-\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)(2n+1)p^2q^2Z_1(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}$$

$$-\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)(2n+1)p^2q^2\big(Z_1(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}$$

$$+\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)(2n+1)p^2q^2\big(Z_1(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\cdot\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\Bigg)$$

$$+a\Bigg(\frac{j(2n+1)pq^2\pi}{j^2-p^2}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{m=0}^{\infty}\left(Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}-Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\right)$$

$$-\frac{pq^2}{\pi}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)Z_2^*(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}$$

$$-\frac{pq^2}{\pi}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)Z_2^*(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}$$

$$+\frac{pq^2}{\pi}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)Z_1^*(p,j)Z_2(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}$$

$$+\frac{pq^2}{\pi}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)Z_2^*(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}$$

$$-\frac{pq^2}{\pi}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)Z_1^*(p,j)Z_2(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\Bigg)\widetilde{V}(p,q,t)$$

$$+(c+at)\Bigg[\frac{(2n+1)pq^2j}{j^2-p^2}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{m=0}^{\infty}\left(Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}-Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\right)$$

$$
\begin{aligned}
&-\frac{2pq^2}{\pi}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)Z_2^*(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\\
&+\frac{2pq^2}{\pi}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)Z_1^*(p,j)Z_1(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\\
&+\frac{pq^2}{\pi}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)Z_2^*(p,j)Z_2(k,q)\frac{\sin(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\\
&-\frac{pq^2}{\pi}\sum_{\substack{p=1\\p\neq j}}^{\infty}\sum_{\substack{q=1\\q\neq k}}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)Z_1^*(p,j)Z_2(k,q)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\left.\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\right)\widetilde{V}_t(p,q,t)\\
&=\frac{\Gamma_1 gL_x^2L_y^2}{j\pi k\pi\left(1+\Gamma_1H_1^*(m,n,t)\right)}\left(-\cos k\pi+\cos\frac{k\pi y_0}{L_y}\right)\left(-\cos j\pi+\cos\frac{j\pi}{L_x}\left(x_0+ct+\frac{1}{2}at^2\right)\right)
\end{aligned}
\tag{72}
$$

where

$$
\begin{aligned}
H_1^*(m,n,t)=\frac{1}{Q_M}&\left\{\frac{L_xL_y}{16}-\frac{L_xk^2}{2}\sum_{m=0}^{\infty}(2m+1)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1}\right.\\
&+\frac{L_x^2L_y}{4\pi^2}\sum_{n=0}^{\infty}(2n+1)Z_1(j)\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\\
&\left.-\frac{jq^2L_y\pi}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(2n+1)(2n+1)Z_1(j)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\right\}
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
H_2^*(m,n,t)=\frac{c+at}{Q_M}&\left\{\frac{j^2L_y}{2\pi}\sum_{m=0}^{\infty}Z_1^*(j)\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\right.\\
&\left.-\frac{2j^2k^2}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}Z_1^*(j)Z_1^*(k)(2m+1)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\right\}
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
H_3^*(m,n,t)=\frac{1}{Q_M}&\left\{(c+at)^2\left[\left(-\frac{j^2\pi^2L_y}{16L_x}\right)+\frac{j^2k^2\pi^2}{4L_x}\sum_{m=0}^{\infty}(2m+1)Z_1(k)\frac{\sin(2m+1)\pi y_0}{2m+1}\right.\right.\\
&+\frac{(2n+1)L_y}{4}\sum_{n=0}^{\infty}(2n+1)Z_1(j)\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}+\frac{2(2n+1)j^2}{\pi}\cdot\\
&\left.\sum_{n=0}^{\infty}(2n+1)Z_1(j)\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}+\frac{2k^2\pi}{2}\sum_{m=0}^{\infty}(2m+1)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1}\right]\\
&+a\left(\frac{j^2L_y}{2\pi}\sum_{n=0}^{\infty}(2n+1)Z_1(j)\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\right.\\
&\left.\left.-\frac{2j^4k^2}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(2m+1)Z_1(j)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\right)\right\}
\end{aligned}
\tag{75}
$$

$$Z_1(j)=\left[\frac{(-1)^j\cos(2n+1)\pi L_x-1}{[(2n+1)L_x]^2-4j^2}-\frac{\cos(2n+1)\pi L_x-1}{[(2n+1)L_x]^2}\right] \tag{76}$$

$$Z_1^*(j)=\left[\frac{2j\{(-1)^{2j}\cos(2n+1)\pi L_x-1\}}{[(2n+1)L_x]^2-4j^2}\right] \tag{77}$$

$$Z_1(k)=\left[\frac{(-1)^j\cos(2m+1)\pi L_x-1}{[(2m+1)L_x]^2-4k^2}-\frac{\cos(2m+1)\pi L_x-1}{[(2m+1)L_x]^2}\right] \tag{78}$$

Next, we seek the modified frequency corresponding to the frequency of the free system due to the presence of the moving distributed mass. An equivalent free system operator defined by the modified frequency then replaces equation (72). To this end, we set the right hand of (72) to zero and consider a parameter $\eta_1^0<1$ for any arbitrary mass ratio Γ_1 defined as

$$\eta_1^0=\frac{\Gamma_1}{1+\Gamma_1} \tag{79}$$

Evidently,

$$\Gamma_1=\eta_1^0+(\eta_1^0)^2+(\eta_1^0)^3+\ldots \tag{80}$$

which to order one gives

$$\Gamma_1=\eta_1^0+O(\eta_1^0)^2 \tag{81}$$

Now

$$\frac{1}{1+\Gamma_1 H_1^*}=\left(1+\Gamma_1 H_1^*\right)^{-1}=1-\Gamma_1 H_1^*+\Gamma_1^2 H_1^{*2}-\Gamma_1^3 H_1^{*3}+\ldots \tag{82}$$

A case corresponding to the case in which the inertia effect of the mass of the system is regarded as negligible is obtained when we set $\eta_1^0=0$ in equation (72). In such a case, the solution to (72) can be written in the form

Whenever

$$\left|\Gamma_1 H_1^*\right|<1 \tag{83}$$

Therefore

$$\frac{\Gamma_1 H_2^*}{1+\Gamma_1 H_1^*}=\Gamma_1 H_2^*\left(1-\Gamma_1 H_1^*+\Gamma_1^2 H_1^{*2}+\ldots\right)=\Gamma_1 H_1^*+O\left(\Gamma_1^2\right) \tag{84}$$

$$\frac{\delta_{SMF}^2 + \Gamma_1 H_3^*}{1 + \Gamma_1 H_1^*} = \left(\delta_{SMF}^2 + \Gamma_1 H_3^*\right)\left(1 - \Gamma_1 H_1^* + \Gamma_1^2 H_1^{*2} + \ldots\right) = \delta_{SMF}^2 + \Gamma_1 H_3^* - \delta_{SMF}^2 \Gamma_1 H_1^* + O\left(\Gamma_1^2\right) \quad (85)$$

$$\frac{\Gamma_1 g L_x^2 L_y^2}{1 + \Gamma_1 H_1^*} = \Gamma_1 g L_x^2 L_y^2 \left(1 - \Gamma_1 H_1^* + \Gamma_1^2 H_1^{*2} + \ldots\right) = \Gamma_1 g L_x^2 L_y^2 + O\left(\Gamma_1^2\right) \quad (86)$$

A case corresponding to the case in which the inertia effect of the mass of the system is regarded as negligible is obtained when we set $\eta_1^0 = 0$ in equation (72). In such a case, the solution to (72) can be written in the form

$$\widetilde{V}\left(j,k,t\right) = C_{MM}^0 \cos\left[\delta_{SMF} t - \phi_{MM}\right] \quad (87)$$

where C_{MM}^0 and ϕ_{MM} are constants.

Furthermore for any arbitrary mass ratio Γ_1 there is always $\eta_1^0 < 1$, the Struble's technique requires that the solution of the homogeneous part of equation (72) be written in an asymptotic form, namely

$$\widetilde{V}\left(j,k,t\right) = D^*(j,k,t)\cos\left[\delta_{SMF} t - \phi(j,k,t)\right] + \eta_1^0 \widetilde{V}\left(j,k,t\right) + O\left(\eta_1^0\right)^2 \quad (88)$$

In order to obtain the modified frequency, equation (88) and its derivatives are substituted into the homogeneous part of equation (72). Therefore, we extract only the variational part of the equations describing the behaviour of $D^*(j,k,t)$ and $\phi(j,k,t)$ during the motion of the distributed mass. The modified frequency is then obtained from these variational equations. Thus, substituting (88) and its derivatives into the homogeneous part of equation (72) and taking into account equation (81) and retaining terms to η_1^0, one obtains

$$\begin{aligned}
&2D^*(j,k,t)\delta_{SMF}\,\dot{\phi}\left(j,k,t\right)\cos\left[\delta_{SMF} t - \phi\left(j,k,t\right)\right] - 2\dot{D}^*(j,k,t)\delta_{SMF}\sin\left[\delta_{SMF} t - \phi\left(j,k,t\right)\right] \\
&- \frac{\eta_1^0 c\,\delta_{SMF}}{Q_M}\left\{ -\frac{j^2 L_y}{2\pi}\sum_{n=0}^{\infty} Z_1^*(j)\frac{\sin(2n+1)\pi\left(x_0 + ct + \frac{1}{2}at^2\right)}{2n+1}\right. \\
&\left. -\frac{4j^2}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Z_1(k) Z_1^*(j)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\sin(2n+1)\pi\left(x_0 + ct + \frac{1}{2}at^2\right)}{2n+1}\right\} D^*(j,k,t)\cdot \\
&\cos\left[\delta_{SMF} t - \phi\left(j,k,t\right)\right] + \frac{\eta_1^0 c}{Q_M}\left\{\frac{j^2\pi^2 L_y}{16L_x} + \frac{L_y}{4}\sum_{n=0}^{\infty}(2n+1)Z_1(j)\frac{\cos(2n+1)\pi\left(x_0 + ct + \frac{1}{2}at^2\right)}{2n+1}\right. \\
&\left. + j^2k^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)(2n+1)Z_1(j)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0 + ct + \frac{1}{2}at^2\right)}{2n+1}\right\} \\
&D^*(j,k,t)\cos\left[\delta_{SMF} t - \phi\left(j,k,t\right)\right] - \frac{\eta_1^0\delta_{SMF}}{Q_M}\left\{\frac{L_x L_y}{16} + \frac{k^2 L_x}{2}\sum_{n=0}^{\infty}(2m+1)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1}\right. \\
&+ \frac{k^2 L_x^2 L_y\pi}{8\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)(2n+1)Z_1(j)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0 + ct + \frac{1}{2}at^2\right)}{2n+1} \\
&\left. + \frac{j^2k^2 L_y\pi}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)(2n+1)Z_1(j)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1}\cdot\frac{\cos(2n+1)\pi\left(x_0 + ct + \frac{1}{2}at^2\right)}{2n+1}\right\}\cdot \\
&D^*(j,k,t)\cos\left[\delta_{SMF} t - \phi\left(j,k,t\right)\right] = 0
\end{aligned} \quad (89)$$

$$\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\cos\left[\delta_{SMF}t-\phi(j,k,t)\right]=\frac{1}{2(2n+1)}\left\{\cos\left[(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)\right.\right.$$
$$\left.\left.+\delta_{SMF}t-\phi(j,k,t)\right]+\cos\left[(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)-\delta_{SMF}t+\phi(j,k,t)\right]\right\}$$
$$\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\sin\left[\delta_{SMF}t-\phi(j,k,t)\right]=\frac{1}{2(2n+1)}\left\{\sin\left[(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)\right.\right.$$
$$\left.+\delta_{SMF}t-\phi(j,k,t)\right]-\sin\left[(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)-\delta_{SMF}t+\phi(j,k,t)\right]$$
$$\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\sin\left[\delta_{SMF}t-\phi(j,k,t)\right]=\frac{1}{2(2n+1)}\left\{\cos\left[(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)\right.\right.$$
$$\left.\left.-\delta_{SMF}t+\phi(j,k,t)\right]-\cos\left[(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)+\delta_{SMF}t-\phi(j,k,t)\right]\right\}$$
$$\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\cos\left[\delta_{SMF}t-\phi(j,k,t)\right]=\frac{1}{2(2n+1)}\left\{\sin\left[(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)\right.\right. \quad (90)$$
$$\left.\left.-\delta_{SMF}t+\phi(j,k,t)\right]-\sin\left[(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)-\delta_{SMF}t+\phi(j,k,t)\right]\right\}$$

Neglecting terms that do not contribute to the variational equations, equation (89) reduces to

$$\begin{aligned}&2D^*(j,k,t)\dot{\phi}(j,k,t)\cos\left[\delta_{SMF}t-\phi(j,k,t)\right]-2\dot{D}^*(j,k,t)\delta_{SMF}\sin\left[\delta_{SMF}t-\phi(j,k,t)\right]\\&+\frac{\eta_1^0c^2}{Q_M}\left[-\frac{j^2\pi^2L_y}{16L_x}+\frac{j^2\pi^2}{4L_x}\sum_{m=0}^{\infty}k^2(2m+1)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1}\right]D^*(j,k,t)\cos\left[\delta_{SMF}t-\phi(j,k,t)\right]\\&-\frac{\eta_1^0\delta_{SMF}}{Q_M}\left[-\frac{L_xL_y}{16}-\frac{L_x}{2}\sum_{m=0}^{\infty}k^2(2m+1)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1}\right]D^*(j,k,t)\cos\left[\delta_{SMF}t-\phi(j,k,t)\right]\end{aligned} \quad (91)$$

Equating the coefficients of the $Cos\left[\delta_{SMF}t-\phi(j,k,t)\right]$ and $Sin\left[\delta_{SMF}t-\phi(j,k,t)\right]$ to zero in equation (91), we obtain

$$2\delta_{SMF}\dot{\phi}(j,k,t)-\eta_1^0\rho_Q\left\{-\rho_1(j,L)+\frac{1}{4}\rho_A(m,y_0)-\delta_{SMF}\left[\frac{L_xL_y}{16}-\frac{1}{2}\rho_B(m,y_0)\right]\right\}=0 \quad (92)$$

and

$$-2\delta_{SMF}\dot{D}^*(j,k,t)=0 \quad (93)$$

respectively as the variational equations of the problem describing the behaviour of $D^*(j,k,t)$ and $\phi(j,k,t)$ where

$$\rho_Q=\frac{1}{Q_M} \quad (94)$$

$$\rho_1(j,L)=\frac{c^2j^2\pi^2L_y}{L_x} \quad (95)$$

$$\rho_A(m,y_0)=\frac{j^2\pi^2}{L_x}\sum_{m=0}^{\infty}k^2(2m+1)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1} \quad (96)$$

$$\rho_B(m,y_0)=L_x\sum_{m=0}^{\infty}k^2(2m+1)Z_1(k)\frac{\cos(2m+1)\pi y_0}{2m+1} \quad (97)$$

$$\phi(j,k,t)=\frac{\eta_1^0\rho_Q}{2}\left\{\frac{\frac{1}{4}\rho_A(m,y_0)-\rho_1(j,L)}{\delta_{SMF}}+\frac{L_xL_y}{16}-\frac{1}{2}\rho_B(m,y_0)\right\}t+\phi_{MM} \tag{98}$$

and

$$D(j,k,t)=C_{MM}^0 \tag{99}$$

$$\tilde{V}(j,k,t)=C_{MM}^0\cos\left[\delta_{SMF}t-\frac{\eta_1^0\rho_Q}{2}\left\{\frac{\frac{1}{4}\rho_A(m,y_0)-\rho_1(j,L)}{\delta_{SMF}}+\frac{L_xL_y-8\rho_B(m,y_0)}{16}\right\}t-\phi_{MM}\right. \tag{100}$$

which after some rearrangement leads to

$$\tilde{V}(j,k,t)=C_{MM}^0\cos\left[\delta_{SMM}t-\phi_{MM}\right] \tag{101}$$

where

$$\delta_{SMM}=\delta_{SMF}\left\{1-\frac{\eta_1^0\rho_Q}{2}\left[\frac{\frac{1}{4}\rho_A(m,y_0)-\rho_1(j,L)}{\delta_{SMF}^2}+\frac{L_xL_y-8\rho_B(m,y_0)}{16\delta_{SMF}}\right]\right\} \tag{102}$$

represent the modified natural frequency representing the frequency of the free system due to the presence of the moving distributed mass. We now replace equation (72) by the equivalent free system defined by the modified frequency δ_{SMM}. Thus, neglecting terms of $O\left(\eta_1^0\right)^2$, the homogeneous part of equation (72) can be written as

$$\tilde{V}_{tt}(j,k,t)+\delta_{SMM}^2\tilde{V}(j,k,t)=0 \tag{103}$$

In view of equation (103), the entire equation (72) reduces to

$$\tilde{V}_{tt}(j,k,t)+\delta_{SMM}^2\tilde{V}(j,k,t)=\frac{\eta_1^0gL_xL_y}{j\pi k\pi}\left\{\left(-\cos k\pi+\cos\frac{k\pi y_0}{L_y}\right)\left(-\cos j\pi+\cos\frac{j\pi}{L_x}\left(x_0+ct+\tfrac{1}{2}at^2\right)\right)\right\} \tag{104}$$

which when solved in conjunction with the initial conditions yields

$$\begin{aligned}
V(x,y,t)=&\frac{4}{L_xL_y}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\eta_1^0L_xL_yg\sqrt{\pi}}{2\delta_{SMM}\sqrt{2a}}\left\{\sin\delta_{SMM}t\left(\sin\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)\right.\right.\\
&+\cos\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)+\sin\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)+\cos\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)\\
&-\sin\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2}{\sqrt{2\pi a}}\right)-\cos\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2}{\sqrt{2\pi a}}\right)-\sin\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1}{\sqrt{2\pi a}}\right)\\
&\left.-\cos\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1}{\sqrt{2\pi a}}\right)-\frac{1}{2\delta_{SMM}}\left[\sin\left(j\pi+\delta_{SMM}t\right)-\sin\left(j\pi-\delta_{SMM}t\right)\right]\right)\\
&-\cos\delta_{SMM}t\left(-\sin\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)+\cos\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)\right.\\
&+\sin\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)-\cos\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)+\sin\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1}{\sqrt{2\pi a}}\right)\\
&-\cos\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1}{\sqrt{2\pi a}}\right)-\sin\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2}{\sqrt{2\pi a}}\right)+\cos\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2}{\sqrt{2\pi a}}\right)\\
&\left.\left.+\frac{1}{\delta_{SMM}}\cos j\pi+\frac{1}{2\delta_{SMM}}\left[\cos\left(j\pi-\delta_{SMM}t\right)-\cos\left(j\pi+\delta_{SMM}t\right)\right]\right)\right\}P_{MK}\left\{\sin\frac{j\pi x}{L_x}\sin\frac{k\pi y}{L_y}\right\}
\end{aligned} \tag{105}$$

which represents the transverse displacement response to travelling partially distributed masses moving at variable velocities of a simply supported isotropic rectangular plate resting on bi-parametric elastic foundation.

4 Discussion of the Analytical Solutions

It is important to establish conditions under which resonance occurs, since the deflection of an elastic plate may grow without bound. Equation (71) shows that the simply supported isotropic rectangular plate resting on Pasternak elastic foundation and traversed by moving distributed force reaches the state of resonance whenever

$$\delta_{SMF} = \frac{j\pi v_c}{L_x}, \qquad \delta_{SMF} = \frac{j\pi v_c}{L_x} + 2at_0 \tag{106}$$

while equation (105) indicates that the same plate under the action of moving distributed mass will experience resonance effect whenever

$$\delta_{SMM} = \frac{j\pi v_c}{L_x}, \qquad \delta_{SMM} = \frac{j\pi v_c}{L_x} + 2at_0 \tag{107}$$

Evidently,

$$\delta_{SMM} = \delta_{SMF}\left\{1 - \frac{\eta_1^0 \rho_Q}{2}\left[\frac{\frac{1}{4}\rho_A(m, y_0) - \rho_1(j, L)}{\delta_{SMF}^2} + \frac{L_x L_y - 8\rho_B(m, y_0)}{16\delta_{SMF}}\right]\right\} = \frac{j\pi v_c}{L_x} \tag{108}$$

Equations (106) and (107) show that for the same natural frequency, the critical velocity for the system consisting of a simply supported isotropic rectangular plate resting on an elastic foundation and traversed by partially distributed moving forces moving with non-uniform velocity is greater than that of the moving mass problem. Hence, for the same natural frequency, resonance is reached earlier in the moving distributed mass system than in the moving distributed force system.

5 Results and Discussion

In order to illustrate the foregoing analysis, an isotropic rectangular plate of lengths $L_x = 4.57m$ and $L_y = 9.14m$ along is considered. The mass per unit length $\mu = 2758.291 kg/m$, modulus of elasticity $E = 2.109 \times 10^9 N/m^2$, moment of inertia $I = 2.87698 \times 10^{-3} m^4$, the plate thickness $h = 0.35m$ and bending rigidity $D = 1000$ is considered. The values of foundation stiffness K is varied between $0N/m^3$ and $400000N/m^3$, the values of axial forces N_x and N_y varied between $0N$ and $2.0 \times 10^8 N$, the shear modulus G is varied between $0N/m$ and $3.0 \times 10^7 N/m$. In Fig. 2, the transverse displacement response of a simply supported isotropic rectangular plate under the action of partially distributed forces moving at variable velocity for various values of foundation stiffness K and fixed values of axial force $N_x = 200000$, shear modulus $G = 300000$ and rotatory inertia correction factor $R^0 = 0.5$. The figure shows that as the foundation stiffness increases, the response amplitude of the rectangular plate decreases. Similar results are obtained when the simply supported plate is subjected to partially distributed masses travelling at variable velocity as shown in Fig. 6. For various travelling time t, the deflection profile of the rectangular plate for various values of axial force N_x and for fixed values of foundation stiffness $K = 40000$, $G = 300000$ and rotatory inertia correction factor $R^0 = 0.5$ is shown in Fig. 3. It is observed that higher values of axial force N_x reduce the deflection profile of the vibrating plate. The same behaviour characterizes the deflection of simply supported rectangular plate under the action of distributed masses moving at variable velocity for various values of axial force N_x as shown in Fig. 7. Also Fig. 4 depict the transverse displacement response of simply supported rectangular plate to partially distributed forces travelling at variable velocity for various values of shear modulus G and for fixed values of foundation stiffness $K = 40000$, axial force $N_x = 200000$ and rotatory inertia correction factor $R^0 = 0.5$. The figures clearly show that the response amplitude of the simply supported isotropic rectangular plate under the action of partially distributed forces travelling at variable velocity decrease with increase in the values of shear modulus G. Similar results are obtained when the simply supported isotropic rectangular plate subjected to a partially distributed masses travelling at variable velocity as shown in Fig. 8. Fig. 5 shows that for various values of rotatory inertia correction factor R^0 and fixed values of foundation stiffness $K = 40000$, axial force $N_x = 200000$ and shear modulus $G = 300000$, higher values of rotatory inertia correction factor reduce the deflection profile of the vibrating plate of simply supported rectangular plate to partially distributed forces travelling at variable velocity. The same behaviour characterizes the deflection profile of the simply supported rectangular plate under the action of partially distributed masses moving at variable velocity for various values of rotatory inertia correction factor R^0 as shown in Fig. 9. Furthermore, Fig. 10 shows the comparison of the transverse response of moving force and moving mass cases for simply supported rectangular plate traversed by a moving load travelling at variable velocity for fixed values of foundation stiffness $K = 40000$, axial force $N_x = 200000$, shear modulus $G = 300000$ and rotatory inertia correction factor $R^0 = 0.5$.

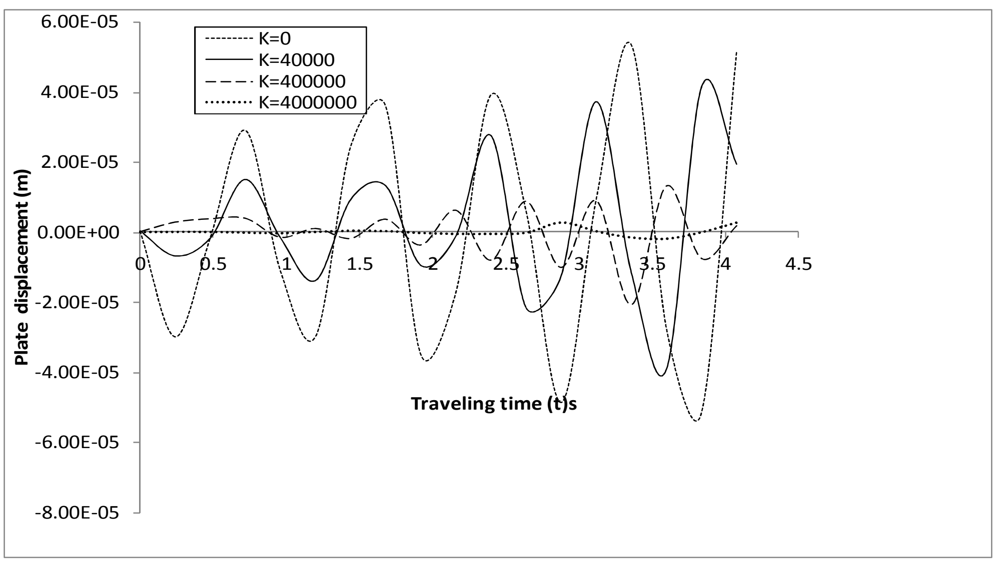

Fig. 2. Transverse displacement of a simply supported rectangular plate under partially distributed forces for various values of K and fixed values of $N_x = 20000$, $G = 300000$ and $R^0 = 0.5$

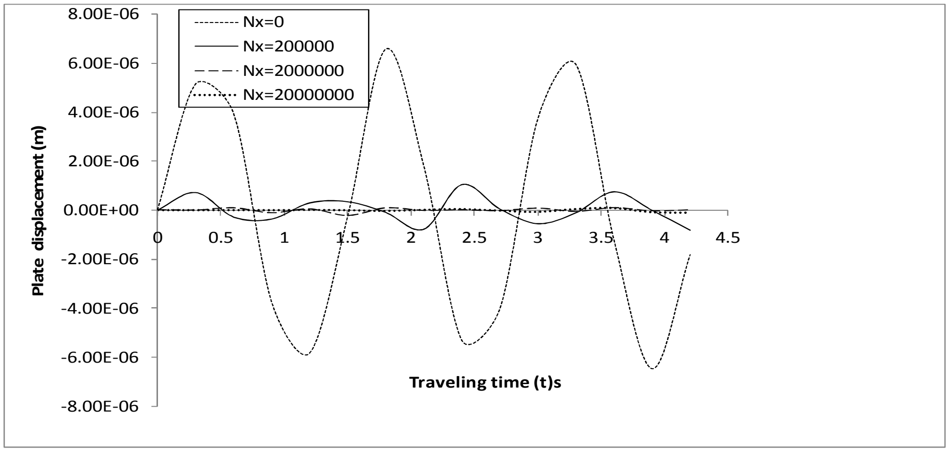

Fig. 3. Deflection profile of a simply supported rectangular plate under partially distributed force for various values of axial force N_x and for fixed values of $K = 40000$, $G = 300000$ and $R^0 = 0.5$

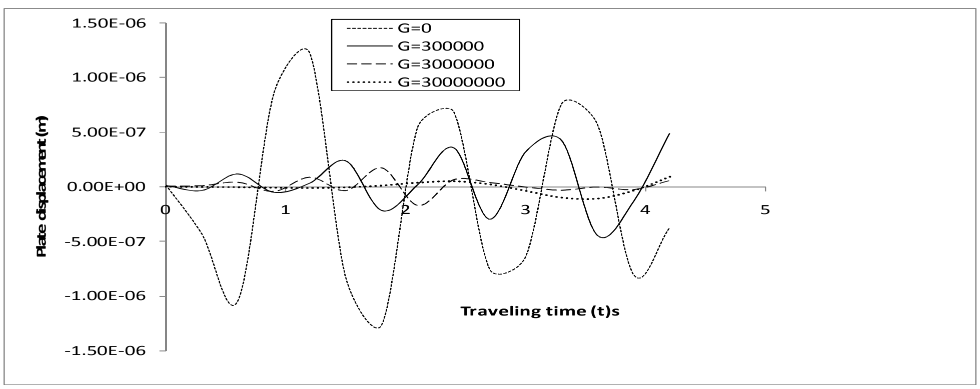

Fig. 4. Response amplitude of simply supported rectangular plate to partially distributed force for various values of G and fixed values of $K = 40000$, $N_x = 20000$ and $R^0 = 0.5$

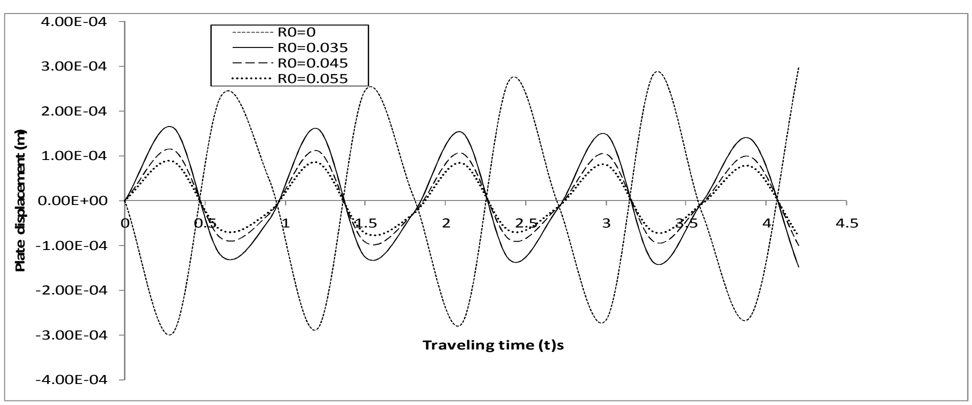

Fig. 5. Transverse displacement of a simply supported rectangular plate under partially distributed forces for various values of R^0 and fixed values of $K = 40000$, $N_x = 20000$ and $G = 300000$

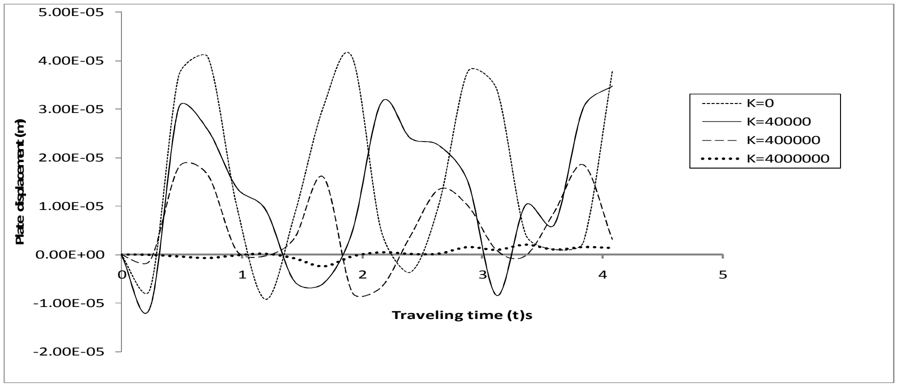

Fig. 6. Deflection profile of a simply supported rectangular plate under partially distributed masses for various values of K and fixed values of $N_x = 20000$, $G = 300000$ and $R^0 = 0.5$

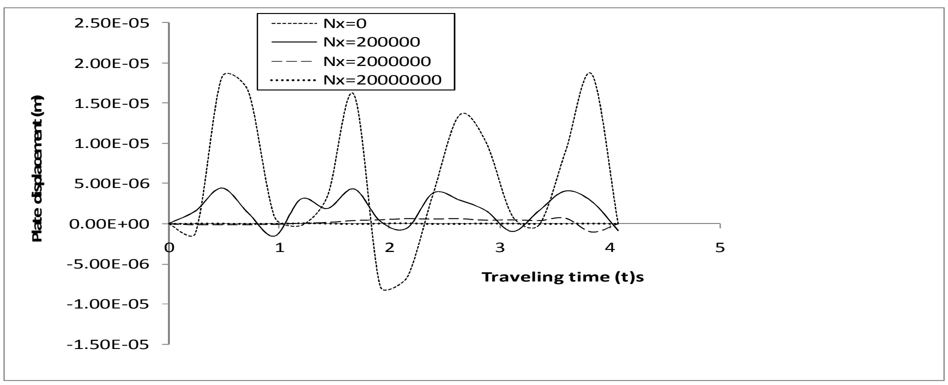

Fig. 7. Response amplitude of a simply supported rectangular plate under partially distributed masses for various values of N_x and for fixed values of $K = 40000$, $G = 300000$ and $R^0 = 0.5$

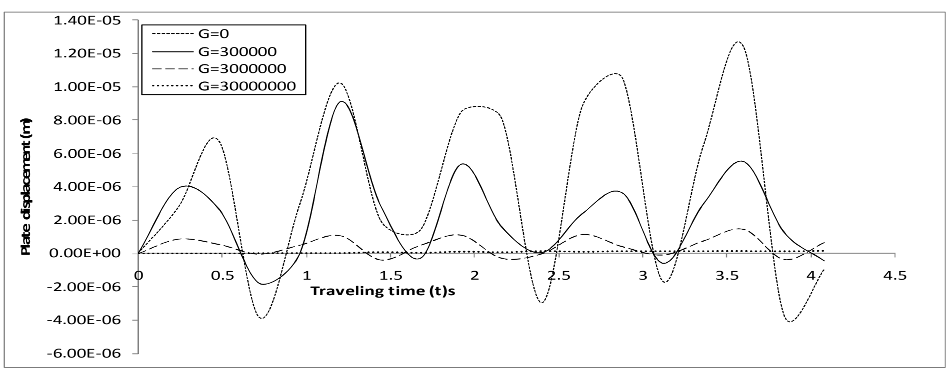

Fig. 8. Transverse displacement of a simply supported rectangular plate to partially distributed masses for various values of G and for fixed values of $K = 40000$, $N_x = 20000$ and $R^0 = 0.5$

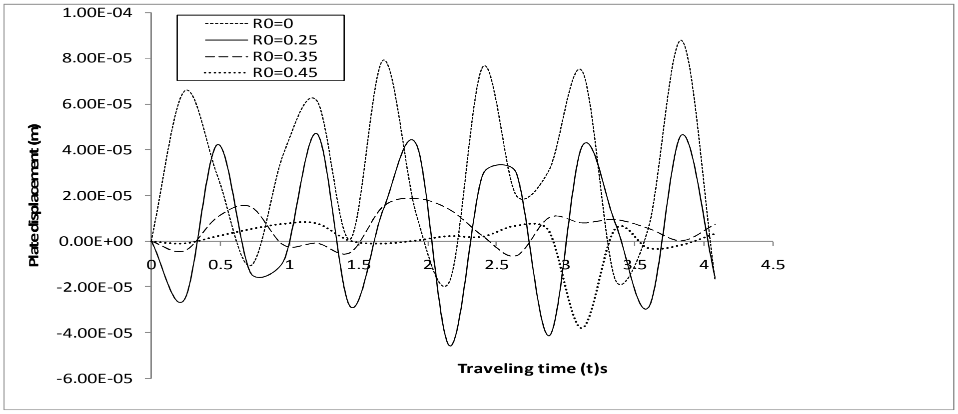

Fig. 9. Deflection profile of a simply supported rectangular plate under partially distributed masses for various values of R^0 and fixed values of $K = 40000$, $N_x = 20000$ and $G = 300000$

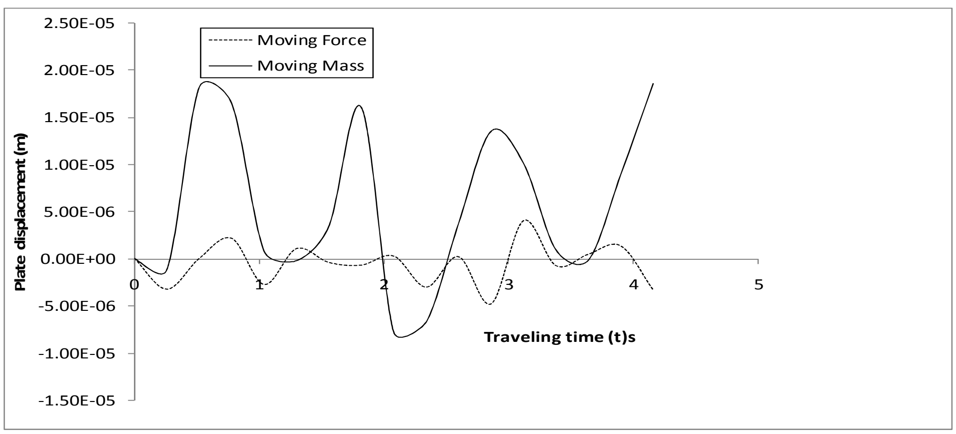

Fig. 10. Comparison of the displacement response of moving force and moving mass cases of a simply supported rectangular plate for fixed values of $K = 40000$, $N_x = 200000$, $G = 300000$ and $R^0 = 0.5$

6 Conclusion

The structure of interest is an isotropic rectangular plate on a Pasternak foundation under the influence of a uniform partially distributed load moving at varying velocities. The governing equation is fourth order partial differential equations with variable and singular coefficients. For this two-dimensional plate problem, the solution technique is based on the modified two-dimensional generalized integral transformation, the expansion of the Heaviside function in series form, a modification of Struble's asymptotic method and then the use of Fresnel sine and cosine integrals. It is shown that increase in pertinent structural parameters such as foundation stiffness, shear modulus, axial force and rotatory inertial correction factor decrease the response amplitude of the plate. For the same natural frequency, the critical velocity for the system consisting of a simply supported isotropic rectangular plate resting on an elastic foundation and traversed by partially distributed moving forces moving with non-uniform velocity is greater than that of the moving mass problem. Hence, for the same natural frequency, resonance is reached earlier in the moving distributed mass system than in the moving distributed force system.

Competing Interests

Authors have declared that no competing interests exist.

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