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# Approximate Solution of an Infectious Disease Model Applying Homotopy Perturbation Method

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**Abstract**

Scientists and engineers have developed the use of Homotopy Perturbation Method (HPM) in non-linear problems since this approach constantly distort the intricate problem being considered into a simple problem, thus making it much less complex to solve. The homotopy perturbation method was initially put forward by He (1999) with further development and improvement (He 2000a, He, 2006). Homotopy, which is as an essential aspect of differential topology involves a coupling of the conventional perturbation method and the homotopy method in topology (He, 2000b). The approach gives an approximate analytical result in series form and has been effectively applied by various academia for various physical systems namely; bifurcation, asymptotology, nonlinear wave equations and Approximate Solution of SIR Infectious Disease Model (Abubakar et al., 2013).

**Keywords:** approximate solution, infectious disease model, homotopy perturbation method

## 1. Model Equations

Considering the following systems of non-linear ordinary differential equation given as;

$$\left.\begin{aligned}
\frac{dS}{dt} &= b + a_1 S_{vc} - \alpha_1 cS - \mu S \\
\frac{dS_v}{dt} &= \alpha_1 cS - (1-\varphi)S_v - \mu S_v \\
\frac{dS_{vc}}{dt} &= (1-\varphi)S_v - (1-e_1)\lambda S_{vc} - a_1 S_{vc} - qS_{vc} - \mu S_{vc} \\
\frac{dS_{vcr}}{dt} &= qS_{vc} - (1-e)\lambda S_{vcr} - \mu S_{vcr} \\
\frac{dS_{ve}}{dt} &= (1-e_1)\lambda S_{vc} + (1-e)\lambda S_{vcr} - \rho_2 S_{ve} - \mu S_{ve} \\
\frac{dI}{dt} &= \rho_2 S_{ve} - (1-\gamma)I - \mu I \\
\frac{dI_t}{dt} &= (1-\gamma)I - d_2 I_t - \mu I_t
\end{aligned}\right\} \quad (1)$$

We let,

$g_1 = (\alpha_1 c + \mu)$ , $g_2 = (1 - \varphi)$, $g_3 = (g_2 + \mu)$,

$g_4 = (1 - e_1)\lambda$, $g_5 = (g_4 + a_1 + q + \mu)$ , $g_6 = (1 - e)\lambda$, $g_7 = (g_6 + \mu)$,

$g_8 = (\rho_2 + \mu)$, $g_9 = (1 - \gamma)$, $g_{10} = (1 - \gamma + \mu)$, $g_{11} = (d_2 + \mu)$

Rewriting (1) in a more compact form, we obtain;

$$\left.\begin{aligned}
\frac{dS}{dt} &= b + a_1 S_{vc} - g_1 S \\
\frac{dS_v}{dt} &= \alpha_1 c S - g_3 S_v \\
\frac{dS_{vc}}{dt} &= g_2 S_v - g_5 S_{vc} \\
\frac{dS_{vcr}}{dt} &= q S_{vc} - g_7 S_{vcr} \\
\frac{dS_{ve}}{dt} &= g_4 S_{vc} + g_6 S_{vcr} - g_8 S_{ve} \\
\frac{dI}{dt} &= \rho_2 S_{ve} - g_{10} I \\
\frac{dI_t}{dt} &= g_9 I - g_{11} I_t
\end{aligned}\right\} \quad (2)$$

### 3. Basic Idea of He's Homotopy Perturbation Method

To demonstrate the basic idea of He's homotopy perturbation method, we consider the non linear differential equation, [He, 2000].

$$A(u) - f(r) = 0 \qquad r \in \Omega \qquad (3)$$

Subject to the boundary condition of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \qquad (4)$$

Given that;

$A$: the general differential operator,

$B$: the boundary operator

$f(r)$ ; a known analytical solution and

$\Gamma$: the boundary of the domain $\Omega$ , Taghipour, (2011)

The general operator, A can be divided into two parts viz; L and N in which L is the linear part and the nonlinear part being N. Hence (3) will now become;

$$L(u) + N(u) - f(r) = 0 \qquad r \in \Omega \qquad (5)$$

We shall now construct a homotopy $V(r, p)$ such that

$V(r, p): \Omega \times [0,1] \to R$ satisfing that;

$$H(r, p) = (1 - p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \qquad (6)$$

$P \in [0,1]$, $r \in \Omega$

Or

$$H(r, p) = L(v) - L(u_0) + pL(u_0) + [N(v) - f(r)] = 0 \qquad (7)$$

Where

$L(u)$ is the linear part

$L(u) = L(v) - L(u_0) + pL(u_0)$ and $N(u)$ is the non-linear term.

$N(u) = pN(v)$

$P \in [0,1]$ is an embedding parameter, while $u_0$ is an initial approximation of equation (3) which satisfies the boundary conditions.

Obviously, considering equations(6) and (7), we have

$$H(v,0) = L(v) - L(u_0) = 0 \tag{8}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{9}$$

The changing process of $p$ from zero to unity is just that of $V(r,p)$ from $u_0$ to $u(r)$. In topology, this is called deformation while $L(v) - L(u_0)$, $A(v) - f(r)$ are called homotopy.

According to Homotopy perturbation method (HPM), we can first use the embedding parameter, $p$ as a small parameter and assume solution for equation (6) and (7) which can be expressed as;

$$V = v_0 + pv_1 + p^2 v_2 + \cdots \tag{10}$$

If we let $p =$ unity, an approximate solution of equation (10) can be obtained as;

$$U = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{11}$$

Equation (11) is the analytical solution of (3) by homotopy perturbation method.

He (2003), (2006) makes the following suggestion for convergence of (11)

(a). The second derivative of $N(v)$ wrt $V$ must be small because parameter, $p$ must be relatively large i.e $p \to 1$

(b). The norm of $L^{-1}\frac{\partial N}{\partial V}$ must be smaller than one so that the series converge.

We now apply HPM on the system (3) by assuming the solution as;

$$\left.\begin{aligned}
S &= u_0 + Pu_1 + P^2 u_2 + \cdots \\
S_v &= w_0 + Pw_1 + P^2 w_2 + \cdots \\
S_{vc} &= x_0 + Px_1 + P^2 x_2 + \cdots \\
S_{vcr} &= y_0 + Py_1 + P^2 y_2 + \cdots \\
S_{ve} &= z_0 + Pz_1 + P^2 z_2 + \cdots \\
I &= m_0 + Pm_1 + P^2 m_2 + \cdots \\
I_t &= n_0 + Pn_1 + P^2 n_2 + \cdots
\end{aligned}\right\} \tag{12}$$

From the the first equation of (12),

$$\frac{dS}{dt} = b + a_1 S_{vc} - g_1 S$$

The linear part is

$$\frac{dS}{dt} = 0$$

and the non-linear part is

$$b + a_1 S_{vc} - g_1 S = 0$$

We now apply HPM

$$\Rightarrow (1-P)\frac{dS}{dt} + P\left[\frac{dS}{dt} - b - a_1 S_{vc} + g_1 S\right] = 0$$

Expanding, this gives

$$\frac{dS}{dt} - P\frac{dS}{dt} + P\frac{dS}{dt} - P(b + a_1 S_{vc} - g_1 S) = 0$$

$$\Rightarrow \frac{dS}{dt} - P(b + a_1 S_{vc} - g_1 S) = 0$$

$$\Rightarrow \frac{dS}{dt} - Pb - Pa_1 S_{vc} + Pg_1 S = 0 \tag{13}$$

Substituting the first and third equations of (12) into (13) gives

$(u_0' + Pu_1' + P^2 u_2' + \cdots +) - Pb - Pa_1(x_0 + Px_1 + P^2 x_2 + \cdots)$
$+ Pg_1(u_0 + Pu_1 + P^2 u_2 + \cdots) = 0$

Collecting the coefficient of powers of $P$, we have;

$$\left.\begin{aligned} &P^0: u_0' = 0 \\ &P^1: u_1' - b - a_1 x_0 + g_1 u_0 = 0 \\ &P^2: u_2' - a_1 x_1 + g_1 u_1 = 0 \end{aligned}\right\} \tag{14}$$

Applying the same approach, we have the following ;

$$\left.\begin{aligned} &P^0: w_0' = 0 \\ &P^1: w_1' - \alpha_1 c u_0 + g_3 w_0 = 0 \\ &P^2: w_2' - \alpha_1 c u_1 + g_3 w_1 = 0 \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} &P^0: x_0' = 0 \\ &P^1: x_1' - g_2 w_0 + g_5 x_0 = 0 \\ &P^2: x_2' - g_2 w_1 + g_5 x_1 = 0 \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} &P^0: y_0' = 0 \\ &P^1: y_1' - q x_0 + g_7 y_0 = 0 \\ &P^2: y_2' - q x_1 + g_7 y_1 = 0 \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} &P^0: z_0' = 0 \\ &P^1: z_1' - g_4 x_0 - g_6 y_0 + g_8 z_0 = 0 \\ &P^2: z_2' - g_4 x_1 - g_6 y_1 + g_8 z_1 = 0 \end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned} &P^0: m_0' = 0 \\ &P^1: m_1' - \rho_2 z_0 + g_{10} m_0 = 0 \\ &P^2: m_2' - \rho_2 z_1 + m_1 g_{10} = 0 \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} &P^0: n_0' = 0 \\ &P^1: n_1' - g_9 m_0 + g_{11_9} n_0 = 0 \\ &P^2: n_2' - g_9 m_1 + g_{11} n_1 = 0 \end{aligned}\right\} \tag{20}$$

From the first equation of (14),

$u_0' = 0$

$\frac{du_0}{dt} = 0$

$\Rightarrow du_0 = 0$

Integrating gives us

$\int du_0 = S_0$

$\therefore u_0 = c_0$

Where $c_0$ is constant of integration. Applying the initial condition we have

$u_0(0) = S_0$

$\Rightarrow c_0 = S_0$

$\therefore u_0 = S_0$

Similarly, we have that;

$$\left.\begin{aligned} &\therefore S_{v0} = w_0 \\ &\therefore S_{vc0} = x_0 \\ &\therefore S_{vcr0} = y_0 \\ &\therefore S_{ve0} = z_0 \\ &\therefore I_0 = m_0 \\ &\therefore I_{t_0} = n_0 \end{aligned}\right\} \tag{21}$$

From the second equation of $(14)$,

$u_1' - b - a_1x_0 + g_1u_0 = 0,$

$u_1' = b + a_1x_0 - g_1u_0$

$\Rightarrow \frac{du_1}{dt} = b + a_1x_0 - g_1u_0$

$$\Rightarrow du_1 = (b + a_1x_0 - g_1u_0)dt \tag{22}$$

Substituting the first and third equations of the system $(21)$ into $(22)$ we obtain;

$du_1 = (b + a_1S_{vc0} - g_1S_0)dt$

Integrating with respect to $t$, we have;

$u_1 = (b + a_1S_{vc0} - g_1S_0)t + c_7$

Where $c_7$ is constant of integration. Applying the initial condition we have;

$u_1(0) = 0, \quad \Rightarrow c_7 = 0$

$\therefore u_1 = (b + a_1S_{vc0} - g_1S_0)t$

Similarly, we have that;

$$\left.\begin{aligned}
&\therefore w_1 = (\alpha_1 c S_0 - g_3 S_{v0})t \\
&\therefore x_1 = (g_2 S_{v0} - g_5 S_{vc0})t \\
&\therefore y_1 = (q S_{vc0} - g_7 S_{vcr0})t \\
&\therefore z_1 = (g_4 S_{vc0} + g_6 S_{vcr0} - g_8 S_{ve0})t \\
&\therefore m_1 = (\rho_2 S_{ve0} - g_{10} I_0)t \\
&\therefore n_1 = \left(g_9 I_0 - g_{11_9} I_{t0}\right)t
\end{aligned}\right\} \qquad (23)$$

From the third equation of (14),

$$u_2' - a_1 x_1 + g_1 u_1 = 0$$

$$u_2' = a_1 x_1 - g_1 u_1$$

$$\Rightarrow \frac{du_2}{dt} = a_1 x_1 - g_1 u_1$$

$$\Rightarrow du_2 = (a_1 x_1 - g_1 u_1\,)dt \qquad (24)$$

Substituting the first and third equations of (23) into (24) we obtain;

$$du_2 = [a_1(g_2 S_{v0} - g_5 S_{vc0})t - g_1(b + a_1 S_{vc0} - g_1 S_0)t]dt$$

$$du_2 = [a_1(g_2 S_{v0} - g_5 S_{vc0}) - g_1(b + a_1 S_{vc0} - g_1 S_0)]tdt$$

$$du_2 = [-bg_1 - (a_1 g_1 + a_1 g_5)s_{vc0} + a_1 g_2 S_{v0} + g_1^2 S_0]tdt$$

Integrating both sides with respect to $t$, we have;

$$u_2 = [-bg_1 - (a_1 g_1 + a_1 g_5)s_{vc0} + a_1 g_2 S_{v0} + g_1^2 S_0]\frac{t^2}{2} + c_{14}$$

Where $c_{14}$ is constant of integration. A*pplyin*g the initial condition we have;

$$u_2(0) = 0, \quad \Rightarrow c_{14} = 0$$

$$\left.\begin{aligned}
&\therefore u_2 = [-bg_1 - (a_1 g_1 + a_1 g_5)s_{vc0} + a_1 g_2 S_{v0} + g_1^2 S_0]\frac{t^2}{2} \\
&\text{Similarly, we have that;} \\
&\therefore w_2 = [\alpha_1 bc - (\alpha_1 c g_1 + \alpha_1 c g_3)s_0 + a_1 \alpha_1 c S_{vc0} + g_3^2 S_{v0}]\frac{t^2}{2} \\
&\therefore x_2 = [\alpha_1 g_2 c S_0 - (g_2 g_3 + g_2 g_5)S_{v0} + g_5^2 S_{vc0}]\frac{t^2}{2} \\
&\therefore y_2 = [q g_2 S_{v0} - (q g_5 + q g_7)S_{vc0} + g_7^2 S_{vcr0}]\frac{t^2}{2} \\
&\therefore z_2 = [g_2 g_4 S_{v0} - (g_4 g_5 + g_4 g_8 - q g_6)S_{vc0} - (g_6 g_7 + g_6 g_8)S_{vcr0} + g_8^2 S_{ve0}]\frac{t^2}{2} \\
&\therefore m_2 = [\rho_2 g_4 S_{vc0} + \rho_2 g_6 S_{vcr0} - (\rho_2 g_8 + \rho_2 g_{10})S_{ve0} + g_{10}^2 I_0]\frac{t^2}{2} \\
&\therefore n_2 = [\rho_2 g_9 S_{ve0} - (g_9 g_{10} + g_9 g_{11})I_0 + g_{11}^2 I_{t0}]\frac{t^2}{2}
\end{aligned}\right\} \qquad (25)$$

Substituting the first equations of (21), (23) and (25) into the number one equation of system (12), we obtain;

$S(t) =$

$$S_0 + P(b + a_1 S_{vc0} - g_1 S_0)t + P^2[-bg_1 - (a_1 g_1 + a_1 g_5)s_{vc0} + a_1 g_2 S_{v0} + g_1^2 S_0]\frac{t^2}{2} + \cdots$$

Setting $p = 1,$ we obtain;

$$\left.\begin{aligned}
S(t) &= S_0 + (b + a_1 S_{vc0} - g_1 S_0)t + [-bg_1 - (a_1 g_1 + a_1 g_5)s_{vc0} + a_1 g_2 S_{v0} + g_1^2 S_0]\frac{t^2}{2} + \cdots \\
S_v(t) &= S_{v0} + (\alpha_1 c S_0 - g_3 S_{v0})t + [\alpha_1 bc - (\alpha_1 c g_1 + \alpha_1 c g_3)s_0 + a_1 \alpha_1 c S_{vc0} + g_3^2 S_{v0}]\frac{t^2}{2} + \cdots \\
S_{vc}(t) &= S_{vc0} + (g_2 S_{v0} - g_5 S_{vc0})t + [\alpha_1 g_2 c S_0 - (g_2 g_3 + g_2 g_5)S_{v0} + g_5^2 S_{vc0}]\frac{t^2}{2} + \cdots \\
S_{vcr}(t) &= S_{vcr0} + (q S_{vc0} - g_7 S_{vcr0})t + [q g_2 S_{v0} - (q g_5 + q g_7)S_{vc0} + g_7^2 S_{vcr0}]\frac{t^2}{2} + \cdots \\
S_{ve}(t) &= S_{ve0} + (g_4 S_{vc0} + g_6 S_{vcr0} - g_8 S_{ve0})t + [g_2 g_4 S_{v0} - (g_4 g_5 + g_4 g_8 - q g_6)S_{vc0} - (g_6 g_7 +
\end{aligned}\right\} \quad (26a)$$

$$g_6 g_8)S_{vcr0} + g_8^2 S_{ve0}]\frac{t^2}{2} + \cdots$$

$$\left.\begin{aligned}
I(t) &= I_0 + (\rho_2 S_{ve0} - g_{10} I_0)t + [\rho_2 g_4 S_{vc0} + \rho_2 g_6 S_{vcr0} - (\rho_2 g_8 + \rho_2 g_{10})S_{ve0} + g_{10}^2 I_0]\frac{t^2}{2} + \cdots \\
I_t(t) &= I_{t0} + \left(g_9 I_0 - g_{11_9} I_{t0}\right)t + [\rho_2 g_9 S_{ve0} - (g_9 g_{10} + g_9 g_{11})I_0 + g_{11}^2 I_{t0}]\frac{t^2}{2} + \cdots
\end{aligned}\right\} \quad (26b)$$

Hence, equations $(45)$ to $(51)$ are our model equations in HPM.

## 4. Conclusion

In this paper, we solved some nonlinear time dependent ordinary differential equations analyticall to obtain approximate solutions using Homotopy Perturbation Method. We considered a system of nonlinear ordinary differential equations arising from the developed mathematical model of an infectious disease. We applied He's same approach in handling the model equations when applying Homotopy Perturbation Method (HPM) to obtain approximate solutions. The result shows the efficiency of homotopy perturbation method in solving nonlinear equations.

**Competing Interests Statement**

The authors declare that there are no competing or potential conflicts of interest.

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