

The Theoretical Overview of Generalized Gumbel Type- II Distribution

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Authors' contributions

This work was carried out in collaboration among all authors. Author SQA designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors AA and RT managed the analyses of the study. Author SQA managed the literature searches. All authors read and approved the final manuscript.

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Abstract

A new version of weighted model is introduced by assigning weights to Gumbel Distribution Type II (GDT-II) to make it more useful and flexible in life time data. This paper provides an understanding of some basic statistical properties of (WGDT-II). It includes moments and maximum likelihood estimator. In addition to the above it imparts the insight to distribution function, survival rate, hazard rate and reverse hazard rate. The behavior of PDF, CDF, and Survival rate, Hazard rates and Reverse Hazard rates has been presented through various graphs by setting different values of parameters.

Keywords: Gumbel distribution Type-II; weighted distribution; maximum likelihood estimator; survival function; hazard rate; reverse hazard rate.

1 Introduction

Gumbel distribution is a particular case of Generalized Extreme Value distribution also known as Fisher-Tippett Distribution is named after Emil Julius Gumbel (1891-1966). It has received notable attention over the years, particularly in extreme value analysis of extreme events [1-3]. Going through Pinheiro and Ferrari,

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gives us an overview of the recent developments and applications of the Gumbel distribution [4-6]. It is applicable in predicting the chance that an extreme earthquake ,flood ,cyclone or any other natural disaster will occur [7,8]. The approach of weighted distributions was primarily introduced by Fisher to the study of effect of methods of ascertainment upon estimation of frequencies [9-11]. Rao detected a unified theory of weighted distributions and spotted various practical cases that can be designed by weighted distributions.

Now,a random variable ‘x’ is said to follow Gumbel Distribution Type -II if its Probability density function (pdf) $f(x)$ is given by ;

$$f(x) = ab x^{-a-1}e^{-bx^{-a}}, x > 0, a, b > 0 \tag{1.1}$$

And the Corresponding Cumulative Distribution Function (cdf) $F(x)$ is given by;

$$F(x) = e^{-bx^{-a}}, x > 0, a, b > 0 \tag{1.2}$$

where ‘a’ is a real parameter and ‘b’ is the shape parameter.

2 Weighted Gumbel Distribution Type-II

Weighted distributions had been repeatedly used in research linked to reliability, bio-medicine, meta-analysis, econometrics, survival analysis, renewal processes, physics, ecology and branching processes can be seen in Patil and Ord, Patil and Rao , Gupta and Keating .Now there is no question that before now Gumbel Distribution Type -II is not popularly used in statistical modeling and the reason *may* be its lack of fits in modeling complex data sets. So, researchers of this distribution from all fields in general and statistics & mathematics in particular have been developing new probability distributions from the existing ones. Here one new model is introduced after assigning weights to Gumbel Distribution Type-II called as Weighted Gumbel Distribution Type-II (WGDT-II).

Weighted distribution is introduced by taking weight function as $w(x) = x^c$, then weight class of distribution can be defined as ;

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))}, \quad 0 < x < \infty \tag{2.1}$$

Where $w(x)$ be a non negative function

To define weighted Gumbel Distribution Type-II formally, if $X \sim$ WGDT-II , then pdf of x is given by;

$$g_w(x; a, b) = \frac{x^c f(x)}{E(x^c)} \tag{2.2}$$

$$\begin{aligned} \text{Now, } E(x^c) &= \int_0^\infty x^c f(x) dx \\ &= \int_0^\infty abx^c x^{-a-1} e^{-bx^{-a}} dx \end{aligned}$$

$$\begin{aligned} \text{Substitute } bx^{-a} &= z \\ E(x^c) &= b^{c/a} \Gamma(1 - c/a) \end{aligned} \tag{2.3}$$

Therefore, substituting value of (2.3) in (2.2), we get;

$$g_w(x; a, b, c) = \frac{abx^{c-a-1}e^{-bx^{-a}}}{b^{c/a}\Gamma(1 - c/a)}, 0 < x < \infty \tag{2.4}$$

$g_w(x; a, b, c)$, in the above equation gives the Probability density function (pdf) of WGD- II .

The Corresponding Cumulative Distribution Function (cdf) of WGD- II is given by;

$$G_w(x; a, b, c) = 1 - \frac{\gamma(1-c/a)}{\Gamma(1-c/a)}; a, b, c > 0; 0 < x < \infty \quad (2.5)$$

Where $\int_0^x e^{s-1} e^{-s} ds$ is the Lower incomplete Gamma function .

The graphs of p.d.f and c.d.f for different values of parameters of WGD- II . are shown in the Figs. 1 and 2 respectively.

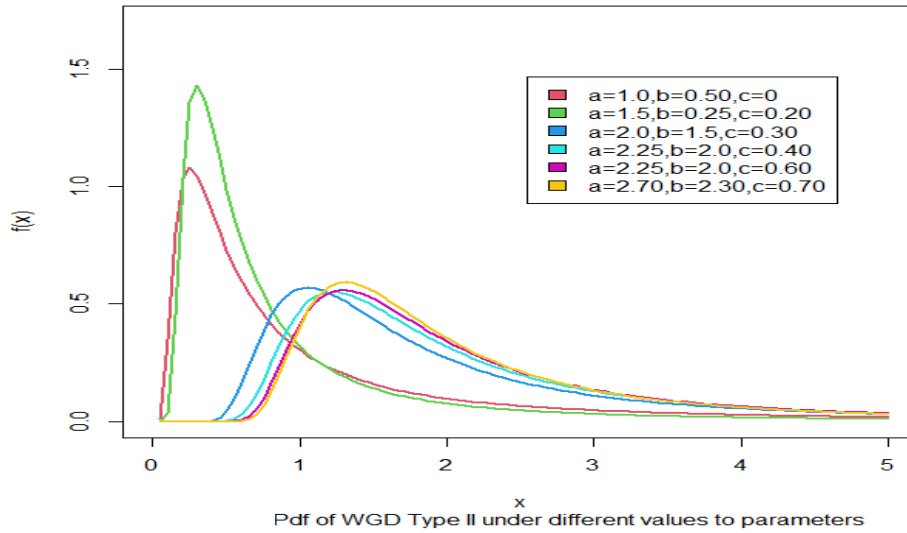


Fig. 1. Probability density function of WGD- II

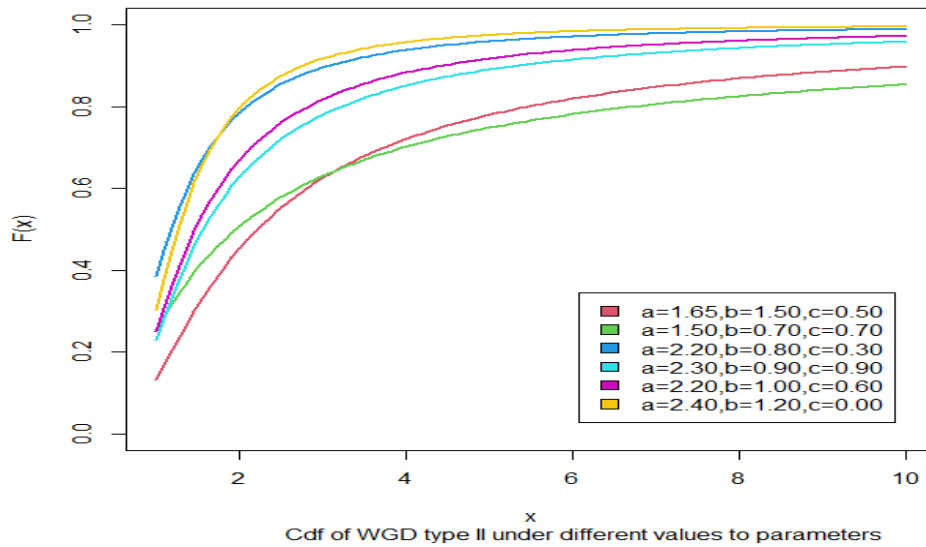


Fig. 2. Cumulative distribution function of WGD- II

3 Reliability Analysis

Here, we have obtained the survival function, hazard rate function and reverse hazard rate function of Weighted Gumbel Distribution Type- II (WGDT- II).

3.1 Survival function

Survival function or reliability function or survivor function is defined as the assessment of probability of longevity beyond the specific time. Now Survival function $S(x)$ of WGDT- II is given as;

$$S_w(x) = \frac{\gamma(1-c/a, bx^{-a})}{\Gamma(1-c/a)} \tag{3.1}$$

3.2 Hazard rate function

The probability of failure of a system when it has survived till time ‘t’ is known as hazard rate or instantaneous failure rate of a random variable ‘x’. The hazard rate $H(x)$ of WGDT- II is given by;

$$H(x) = \frac{x^{c-a-1}e^{-bx^{-a}}ab}{b^c/a\gamma(1-c/a, bx^{-a})} \tag{3.2}$$

3.3 Reverse hazard rate function

Reverse hazard rate function $RH(x)$ is given by;

$$RH(X) = \frac{x^{c-a-1}e^{-bx^{-a}}ab}{b^c/a[\Gamma(1-c/a)-\gamma(1-c/a, bx^{-a})]} \tag{3.3}$$

The graph of survival function, hazard rate function and reverse hazard rate function is shown in Figs. 3, 4 and 5 respectively.

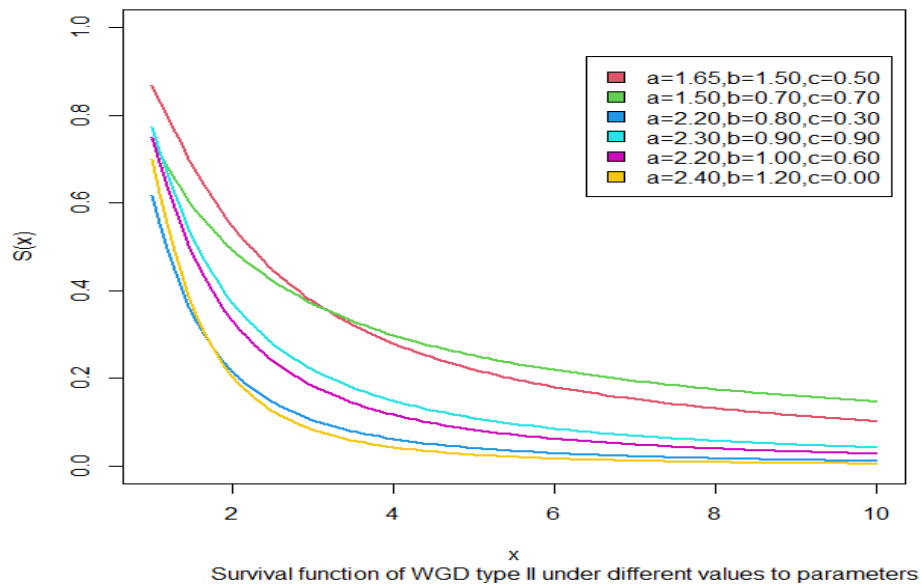


Fig. 3. Survival function of WGDT- II

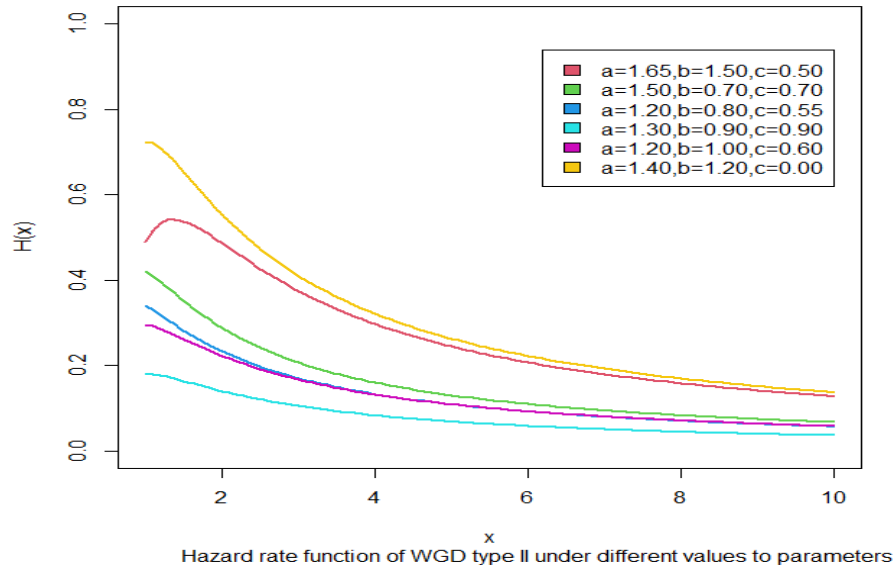


Fig. 4. Hazard rate function of WGD- II

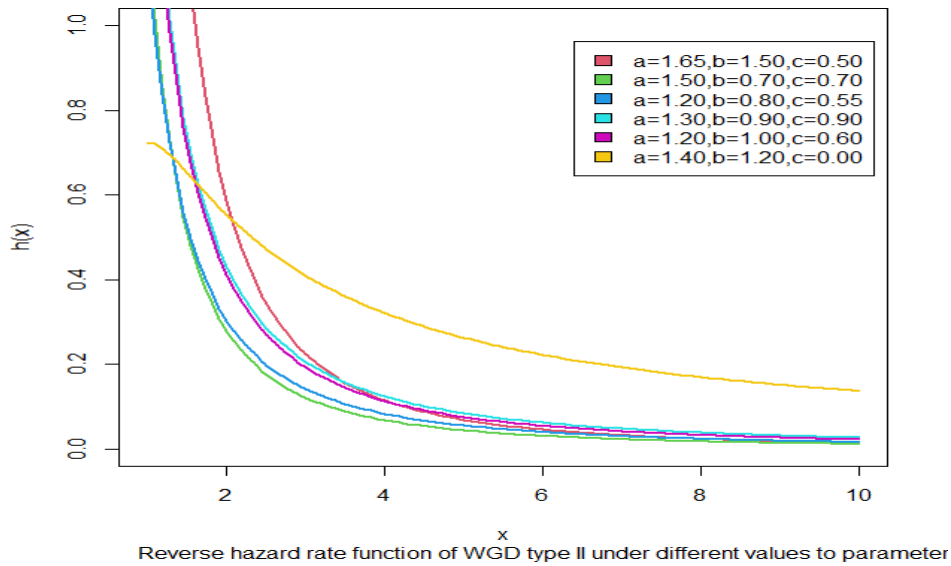


Fig. 5. Reverse hazard rate function of WGD- II

4 Statistical Properties

4.1 Moments of Weighted Gumbel Distribution Type-II

If 'x' is a random variable following WGD- II . Then the r^{th} moment is given by;

$$\mu'_r = E(x^r) = \int_0^\infty x^r g(x; c, a, b) dx$$

Putting the value of $g(x; c, a, b)$ from (2.4), we get;

$$\begin{aligned} &= \int_0^\infty x^r \frac{ab}{b^{c/a}} \frac{x^{c-a-1} e^{-bx^{-a}}}{\Gamma(1-c/a)} dx \\ \mu'_r &= \frac{b^{r/a}}{\Gamma(1-c/a)} \Gamma\left\{1 - \frac{r+c}{a}\right\} \end{aligned} \tag{4.1}$$

Substituting $r = 1, 2, 3, 4$ in the above equation respectively, we get ;

$$\begin{aligned} \text{Mean } \mu'_1 &= \frac{b^{1/a}}{\Gamma(1-c/a)} \Gamma\left[1 - \frac{1+c}{a}\right], \mu'_2 = \frac{b^{2/a}}{\Gamma[1-c/a]} \Gamma\left[1 - \frac{2+c}{a}\right], \mu'_3 = \frac{b^{3/a}}{\Gamma[1-c/a]} \Gamma\left[1 - \frac{3+c}{a}\right], \\ \mu'_4 &= \frac{b^{4/a}}{\Gamma[1-c/a]} \Gamma\left[1 - \frac{4+c}{a}\right] \\ \text{Variance } \sigma^2 = \mu_2 &= \frac{b^{2/a}}{\Gamma(1-\frac{c}{a})^2} \left\{ \Gamma\left(1 - \frac{2+c}{a}\right) \Gamma\left(1 - \frac{c}{a}\right) - \Gamma\left(1 - \frac{1+c}{a}\right)^2 \right\} \end{aligned} \tag{4.2}$$

Now, the moments about mean are derived below by using relationship between moments about mean and moments about origin;

$$\begin{aligned} \mu_2 &= \frac{b^{2/a}}{\Gamma(1-\frac{c}{a})^2} \left\{ \Gamma\left(1 - \frac{2+c}{a}\right) \Gamma\left(1 - \frac{c}{a}\right) - \Gamma\left(1 - \frac{1+c}{a}\right)^2 \right\} \\ \mu_3 &= \frac{b^{2/a}}{\Gamma(1-\frac{c}{a})} \left[\Gamma\left[1 - \frac{3+c}{a}\right] - \frac{3b^{1/a}}{\Gamma(1-c/a)} \Gamma\left[1 - \frac{2+c}{a}\right] + \frac{2}{\Gamma(1-c/a)} \right] \end{aligned} \tag{4.3}$$

$$\mu_4 = \frac{b^{3/a}}{\Gamma(1-\frac{c}{a})^4} \left\{ \Gamma\left(1 - \frac{c}{a}\right)^3 \Gamma\left[1 - \frac{4+c}{a}\right] - 4 b^{1/a} \Gamma\left(1 - \frac{c}{a}\right)^2 \Gamma\left[1 - \frac{3+c}{a}\right] + 6 \Gamma\left(1 - \frac{c}{a}\right)^2 \Gamma\left[1 - \frac{2+c}{a}\right] - 3 \Gamma\left(1 - \frac{c}{a}\right) b^{1/a} \right\} \tag{4.4}$$

The values of standard deviation (S.D), coefficient of Skewness, coefficient of kurtosis, coefficient of variation (C.V), index of dispersion of the above distribution are obtained as under;

a) Standard Deviation

$$\sigma = \sqrt{var} = \frac{b^{1/a}}{\Gamma(1-c/a)} \left[\sqrt{\Gamma\left(1 - \frac{2+c}{a}\right) \Gamma\left(1 - \frac{c}{a}\right) - \Gamma\left(1 - \frac{1+c}{a}\right)^2} \right] \tag{4.5}$$

b) Coefficient of Skewness

$$\sqrt{\beta_1} = \frac{\frac{b^{2/a}}{\Gamma(1-\frac{c}{a})} \left\{ \Gamma\left[1 - \frac{3+c}{a}\right] - \frac{3b^{1/a}}{\Gamma(1-c/a)} \Gamma\left[1 - \frac{2+c}{a}\right] + \frac{2}{\Gamma(1-c/a)} \right\}}{b^{3/a} \left[\frac{\Gamma\left(1 - \frac{c}{a}\right) \Gamma\left[1 - \frac{2+c}{a}\right]}{\Gamma(1-\frac{c}{a})^2} - \Gamma\left(1 - \frac{1+c}{a}\right)^2 \right]^{3/2}} \tag{4.6}$$

c) Coefficient of Kurtosis

$$\beta_2 = \frac{b^{1/a} \left\{ \Gamma\left(1 - \frac{c}{a}\right)^2 \Gamma\left[1 - \frac{4+c}{a}\right] - 4b^{1/a} \Gamma\left(1 - \frac{c}{a}\right) \Gamma\left[1 - \frac{3+c}{a}\right] + 6 \Gamma\left(1 - \frac{c}{a}\right) \Gamma\left[1 - \frac{2+c}{a}\right] - 3b^{1/a} \right\}}{\Gamma\left(1 - \frac{c}{a}\right) \Gamma\left[1 - \frac{2+c}{a}\right]^2 + b^{2/a} \Gamma\left(1 - \frac{1+c}{a}\right) 4 \Gamma\left(1 - \frac{c}{a}\right)^{-1} - 2b^{1/a} \Gamma\left[1 - \frac{2+c}{a}\right] \Gamma\left(1 - \frac{1+c}{a}\right)^2} \tag{4.7}$$

d) Coefficient of variation

$$\frac{\sigma}{\mu_1} = \left[\sqrt{\Gamma\left(1 - \frac{2+c}{a}\right) \Gamma\left(1 - \frac{c}{a}\right) - \Gamma\left(1 - \frac{1+c}{a}\right)^2} \right] \quad (4.8)$$

e) Index of dispersion

$$\frac{\sigma^2}{\mu_1} = \frac{b^{1/a}}{\Gamma(1-c/a)} \Gamma\left(1 - \frac{2+c}{a}\right) \Gamma\left(1 - \frac{c}{a}\right) - \Gamma\left(1 - \frac{1+c}{a}\right)^2 \quad (4.9)$$

Table 1. Brief description of WGDT- II for different values of parameter combinations

A	B	C	Mean	Variance	C.V	Skew.	Kurtosis
8	2	0.9	1.0107	0.0595	0.1571	2.8844	0.1781
7	2.7	1.8	0.9331	0.1523	0.1873	7.0677	4.2139
9.5	3	2.8	0.8703	0.0699	0.1511	5.9056	4.2592
3	3.6	0.4	1.3970	4.4218	0.2139	5.2618	-90.204
6.5	4	0.6	1.1649	0.1262	0.6155	1.8496	-0.6104
4	4.9	0.7	1.3082	1.2451	0.2084	54.369	-176.03
5	16	1.5515	1.5515	0.6922	0.1529	4.8766	6.4748

5 Estimation Methods

5.1 Maximum likelihood estimator

Method of MLE is the most popular technique for deriving estimators of a probability distribution. Let x_1, x_2, \dots, x_i be a random sample from WGDT- II then corresponding likelihood function is given by;

$$\begin{aligned} L(c, a, b|x_i) &= \prod_{i=1}^n g(x_i) \\ &= \frac{(ab)^n}{b^{nc/a} \Gamma(1-c/a)^n} e^{-b \sum_{i=1}^n x_i^{-a}} \prod_{i=1}^n x_i^{c-a-1} \end{aligned} \quad (5.1)$$

The Likelihood function corresponding to (5.1) is obtained as;

$$\text{Log}L(c, a, b|x_i) = n \log a + n \log b - \frac{nc}{a} \log b - n \log \Gamma\left[1 - \frac{c}{a}\right] - b \sum_{i=1}^n x_i^{-a} + (c - a - 1) \log \sum_{i=1}^n x_i \quad (5.2)$$

Taking $\frac{\partial \log L}{\partial b} = 0$ in equation (5.2), The MLE of b can be obtained as;

$$\begin{aligned} \frac{n}{b} - \frac{nc}{ab} - \sum_{i=1}^n x_i^{-a} &= 0 \\ \hat{b} &= \frac{n(a-c) \sum_{i=1}^n x_i^{-a}}{a} \end{aligned} \quad (5.3)$$

Now taking $\frac{\partial \log L}{\partial a} = 0$, and $\frac{\partial \log L}{\partial c} = 0$ in equation (5.2), we obtain the normal equations below;

$$\frac{n}{a} \left[1 + \frac{c}{a} \log b \right] = n \frac{\Gamma\left(\frac{c}{a}\right)}{\Gamma\left[1 - \frac{c}{a}\right]} + \frac{b \log x_i}{\sum_{i=1}^n x_i^a} \quad (5.4)$$

and
$$\Gamma\left[1 - \frac{c}{a}\right] = \frac{\log \sum_{i=1}^n (x_i)^{-\frac{n}{a}} \log b}{n \Gamma\left[-\frac{1}{a}\right]} \quad (5.5)$$

6 Conclusion

In this paper, a new distribution namely Weighted Gumbel Distribution Type-II is introduced. Various statistical properties like mean and variance have been derived. We give precise expressions for survival function $S(x)$, hazard rate function $H(x)$ and reverse hazard rate $RH(x)$. The values of standard deviation (S.D), coefficient of Skewness, coefficient of kurtosis, coefficient of variation (C.V), index of dispersion of the WGDT- II are obtained. The model parameter is estimated by Method of Likelihood Estimation (MLE).

Competing Interests

Authors have declared that no competing interests exist.

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