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The Convolution Sums $\sum_{m<\frac{n}{4}} m\sigma_3(m)\sigma_5(n-4m)$ and $\sum_{m<\frac{n}{4}} m\sigma_5(m)\sigma_3(n-4m)$

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Abstract

In this paper, we obtain the convolution sum formulae of

$$\sum_{m<\frac{n}{4}} m\sigma_3(m)\sigma_5(n-4m) \quad \text{and} \quad \sum_{m<\frac{n}{4}} m\sigma_5(m)\sigma_3(n-4m).$$

Moreover we obtain some identities induced from the above convolution sums and we find a coefficients relation.

Keywords: Divisor functions; Convolution sums

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1 Introduction

For $n \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define the important divisor function and the infinite product sums, which also appear in many areas of number theory:

$$\begin{aligned} \sigma_s(n) = \sum_{d|n} d^s, \qquad \Delta(q) &:= \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \\ A(q) &:= \sum_{n=1}^{\infty} a(n) q^n = \Delta(q^2)^{\frac{1}{2}} = q \prod_{n=1}^{\infty} (1-q^{2n})^{12}, \\ B(q) &:= \sum_{n=1}^{\infty} b(n) q^n = \big(\Delta(q)\Delta(q^2)\big)^{\frac{1}{3}} = q \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^8, \\ C(q) &:= \sum_{n=1}^{\infty} c(n) q^n = \big(\Delta(q)^4\Delta(q^2)\big)^{\frac{1}{6}} = q \prod_{n=1}^{\infty} (1-q^n)^{16} (1-q^{2n})^4, \\ D(q) &:= \sum_{n=1}^{\infty} d(n) q^n = \big(\Delta(q)^2\Delta(q^2)\Delta(q^4)^2\big)^{\frac{1}{6}} = q^2 \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^4 (1-q^{4n})^8, \\ F(q) &:= \sum_{n=1}^{\infty} f(n) q^n = \left(\frac{\Delta(q^4)^4}{\Delta(q)}\right)^{\frac{1}{3}} = q^5 \prod_{n=1}^{\infty} (1-q^n)^{24} (1+q^n)^{32} (1+q^{2n})^{32}. \end{aligned} \tag{1.1}$$

Here we obtain the simple relation between $c(n)$ and $d(n)$, which enables us to express $c(n)$ with $d(n)$:

Theorem 1.1. *Let $n \in \mathbb{N}$. Then we have*

$$c(n) = d(2n) - 32d(n).$$

Let $q \in \mathbb{C}$ be such that $|q| < 1$. The Eisenstein series $L(q)$, $M(q)$, and $N(q)$ are

$$L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \tag{1.2}$$

$$M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{1.3}$$

$$N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \tag{1.4}$$

by ([1], p. 318). Lahiri ([2], p. 149) has derived the following identities from the work of Ramanujan [3]:

$$M^2(q) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \tag{1.5}$$

$$M^3(q) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n + \frac{432000}{691} \sum_{n=1}^{\infty} \tau(n) q^n, \tag{1.6}$$

$$L(q)M(q) = 1 + 720 \sum_{n=1}^{\infty} n\sigma_3(n) q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \tag{1.7}$$

$$N^2(q) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{762048}{691}\sum_{n=1}^{\infty}\tau(n)q^n, \tag{1.8}$$

$$L(q)N(q) = 1 - 1008\sum_{n=1}^{\infty}n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n, \tag{1.9}$$

Specially, we refer to

$$L(q)M(q)N(q) = 1 - \frac{1584}{5}\sum_{n=1}^{\infty}n\sigma_9(n)q^n + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{228096}{3455}\sum_{n=1}^{\infty}\tau(n)q^n \tag{1.10}$$

in ([4], Lemma 3.4). For $e, f, m, n \in \mathbb{N}$ we define

$$I_{e,f}(n) := \sum_{m=1}^{n-1}\sigma_e(m)\sigma_f(n-m). \tag{1.11}$$

Ramanujan [3] and Lahiri [2], [5] have shown that $I_{e,f}$ can be expressed as :

$$\begin{aligned}
I_{1,1}(n) &= \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma_1(n),\\
I_{1,3}(n) &= \frac{7}{80}\sigma_5(n) + \frac{(1-3n)}{24}\sigma_3(n) - \frac{1}{240}\sigma_1(n),\\
I_{1,5}(n) &= \frac{5}{126}\sigma_7(n) + \frac{(1-2n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(n),\\
I_{3,3}(n) &= \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n),\\
I_{1,7}(n) &= \frac{11}{480}\sigma_9(n) + \frac{(2-3n)}{48}\sigma_7(n) - \frac{1}{480}\sigma_1(n),\\
I_{3,5}(n) &= \frac{11}{5040}\sigma_9(n) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(n),\\
I_{1,9}(n) &= \frac{455}{30404}\sigma_{11}(n) + \frac{(5-6n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1(n) - \frac{36}{3455}\tau(n),\\
I_{3,7}(n) &= \frac{91}{110560}\sigma_{11}(n) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(n) + \frac{15}{2764}\tau(n),\\
I_{5,5}(n) &= \frac{65}{174132}\sigma_{11}(n) + \frac{1}{252}\sigma_5(n) - \frac{3}{691}\tau(n),\\
I_{1,11}(n) &= \frac{691}{65520}\sigma_{13}(n) + \frac{(1-n)}{24}\sigma_{11}(n) - \frac{691}{65520}\tau(n),\\
I_{3,9}(n) &= \frac{1}{2640}\sigma_{13}(n) - \frac{1}{240}\sigma_9(n) + \frac{1}{264}\sigma_3(n),\\
I_{5,7}(n) &= \frac{1}{10080}\sigma_{13}(n) + \frac{1}{504}\sigma_7(n) - \frac{1}{480}\sigma_5(n).
\end{aligned}$$

Similarly, for $e, f, m, n \in \mathbb{N}$ we set

$$\begin{aligned}
T_{m,e,f}(n) &:= \sum_{m<\frac{n}{2}} m\sigma_e(m)\sigma_f(n-2m),\\
T_{e,f}(n) &:= \sum_{m<\frac{n}{2}} \sigma_e(m)\sigma_f(n-2m),
\end{aligned} \tag{1.12}$$

and

$$
\begin{aligned}
U_{m,e,f}(n) &:= \sum_{m<\frac{n}{4}} m\sigma_e(m)\sigma_f(n-4m),\\
U_{e,f}(n) &:= \sum_{m<\frac{n}{4}} \sigma_e(m)\sigma_f(n-4m).
\end{aligned} \tag{1.13}
$$

Then in ([6], p. 45–54) we can see that

$$
\begin{aligned}
T_{1,1}(n) &= \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(\frac{n}{2}) + \frac{(1-3n)}{24}\sigma_1(n) + \frac{(1-6n)}{24}\sigma_1(\frac{n}{2}),\\
T_{3,1}(n) &= \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5(\frac{n}{2}) + \frac{(1-3n)}{24}\sigma_3(\frac{n}{2}) - \frac{1}{240}\sigma_1(n),\\
T_{1,3}(n) &= \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5(\frac{n}{2}) + \frac{(2-3n)}{48}\sigma_3(n) - \frac{1}{240}\sigma_1(\frac{n}{2}),\\
T_{5,1}(n) &= \frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7(\frac{n}{2}) + \frac{(1-2n)}{24}\sigma_5(\frac{n}{2}) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}b(n),\\
T_{3,3}(n) &= \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7(\frac{n}{2}) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{2}) + \frac{1}{272}b(n),\\
T_{1,5}(n) &= \frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7(\frac{n}{2}) + \frac{(1-n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(\frac{n}{2}) - \frac{1}{102}b(n),\\
T_{9,1}(n) &= \frac{1}{91212}\sigma_{11}(n) + \frac{31}{2073}\sigma_{11}(\frac{n}{2}) + \frac{(5-6n)}{120}\sigma_9(\frac{n}{2}) + \frac{1}{264}\sigma_1(n)\\
&\quad - \frac{21}{5528}\tau(n) - \frac{282}{3455}\tau(\frac{n}{2}),\\
T_{7,3}(n) &= \frac{1}{331680}\sigma_{11}(n) + \frac{17}{20730}\sigma_{11}(\frac{n}{2}) - \frac{1}{240}\sigma_7(\frac{n}{2}) - \frac{1}{480}\sigma_3(n)\\
&\quad + \frac{23}{11056}\tau(n) + \frac{91}{1382}\tau(\frac{n}{2}),\\
T_{5,5}(n) &= \frac{1}{174132}\sigma_{11}(n) + \frac{16}{43533}\sigma_{11}(\frac{n}{2}) + \frac{1}{504}\sigma_5(n) + \frac{1}{504}\sigma_5(\frac{n}{2})\\
&\quad - \frac{11}{5528}\tau(n) - \frac{88}{691}\tau(\frac{n}{2}),\\
T_{3,7}(n) &= \frac{17}{331680}\sigma_{11}(n) + \frac{8}{10365}\sigma_{11}(\frac{n}{2}) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(\frac{n}{2})\\
&\quad + \frac{91}{22112}\tau(n) + \frac{368}{691}\tau(\frac{n}{2}),\\
T_{1,9}(n) &= \frac{31}{8292}\sigma_{11}(n) + \frac{256}{22803}\sigma_{11}(\frac{n}{2}) + \frac{(5-3n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1(\frac{n}{2})\\
&\quad - \frac{141}{6910}\tau(n) - \frac{2688}{691}\tau(\frac{n}{2})
\end{aligned}
$$

and

$$\begin{aligned}
T_{7,1}(n) &= \frac{1}{14880}\sigma_9(n) + \frac{17}{744}\sigma_9(\frac{n}{2}) + \frac{(2-3n)}{48}\sigma_7(\frac{n}{2}) - \frac{1}{480}\sigma_1(n) \\
&+ \frac{1}{496}c(n) + \frac{2}{31}d(n), \\
T_{5,3}(n) &= \frac{1}{31248}\sigma_9(n) + \frac{1}{465}\sigma_9(\frac{n}{2}) - \frac{1}{240}\sigma_5(\frac{n}{2}) + \frac{1}{504}\sigma_3(n) \\
&- \frac{1}{496}c(n) - \frac{2}{31}d(n), \\
T_{3,5}(n) &= \frac{1}{7440}\sigma_9(n) + \frac{4}{1953}\sigma_9(\frac{n}{2}) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(\frac{n}{2}) \\
&+ \frac{1}{248}c(n) + \frac{4}{31}d(n), \\
T_{1,7}(n) &= \frac{17}{2976}\sigma_9(n) + \frac{8}{465}\sigma_9(\frac{n}{2}) + \frac{(4-3n)}{96}\sigma_7(n) - \frac{1}{480}\sigma_1(\frac{n}{2}) \\
&- \frac{1}{62}c(n) - \frac{16}{31}d(n).
\end{aligned} \tag{1.14}$$

Also we can find

$$T_{m,1,1}(n) = \frac{1}{48}n\sigma_3(n) - \frac{1}{48}n^2\sigma_1(n) + \frac{1}{12}n\sigma_3(\frac{n}{2}) + \left(\frac{1}{48}n - \frac{1}{12}n^2\right)\sigma_1(\frac{n}{2}) \tag{1.15}$$

in ([7], Theorem 4.1) and we can see that

Proposition 1.1. *(See ([8], Theorem 1.1)) Let $n \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned}
T_{m,1,3}(n) &= \frac{1}{1440}\left[n\left\{5\sigma_5(n) + 16\sigma_5(\frac{n}{2}) - 9n\sigma_3(n) - 3\sigma_1(\frac{n}{2})\right\} + 4b(n)\right], \\
T_{m,3,1}(n) &= \frac{1}{720}\left[n\left\{\sigma_5(n) + 20\sigma_5(\frac{n}{2}) - (36n-15)\,\sigma_3(\frac{n}{2})\right\} - b(n)\right],
\end{aligned}$$

(b)

$$\begin{aligned}
T_{m,1,5}(n) &= \frac{1}{17136}\left\{21n\sigma_7(n) + 64n\sigma_7(\frac{n}{2}) - 51n^2\sigma_5(n) + 17n\sigma_1(\frac{n}{2}) + 1632d(n)\right. \\
&\quad \left. + 51c(n) - 21nb(n)\right\}, \\
T_{m,3,3}(n) &= \frac{1}{16320}\left\{2n\sigma_7(n) + 32n\sigma_7(\frac{n}{2}) - 34n\sigma_3(\frac{n}{2}) - 544d(n) - 17c(n)\right. \\
&\quad \left. + 15nb(n)\right\}, \\
T_{m,5,1}(n) &= \frac{1}{22848}\left\{4n\sigma_7(n) + 336n\sigma_7(\frac{n}{2}) - 816n^2\sigma_5(\frac{n}{2}) + 476n\sigma_5(\frac{n}{2}) + 544d(n)\right. \\
&\quad \left. + 17c(n) - 21nb(n)\right\},
\end{aligned}$$

(c)

$$\begin{aligned}
T_{m,1,7}(n) &= \frac{1}{446400}\Big\{255n\sigma_9(n)+768n\sigma_9(\frac{n}{2})-775n^2\sigma_7(n)-465n\sigma_1(\frac{n}{2})\\
&\quad +1240\tau(n)+190464\tau(\frac{n}{2})-23040nd(n)-720nc(n)\Big\},\\
T_{m,3,5}(n) &= \frac{1}{781200}\Big\{21n\sigma_9(n)+320n\sigma_9(\frac{n}{2})+775n\sigma_3(\frac{n}{2})-651\tau(n)-59520\tau(\frac{n}{2})\\
&\quad +20160nd(n)+630nc(n)\},\\
T_{m,5,3}(n) &= \frac{1}{520800}\Big\{5n\sigma_9(n)+336n\sigma_9(\frac{n}{2})-1085n\sigma_5(\frac{n}{2})+310\tau(n)+13888\tau(\frac{n}{2})\\
&\quad -10080nd(n)-315nc(n)\},\\
T_{m,7,1}(n) &= \frac{1}{111600}\Big\{3n\sigma_9(n)+1020n\sigma_9(\frac{n}{2})-3100n^2\sigma_7(\frac{n}{2})+2325n\sigma_7(\frac{n}{2})\\
&\quad -93\tau(n)-2480\tau(\frac{n}{2})+2880nd(n)+90nc(n)\Big\}.
\end{aligned}$$

Furthermore, in ([6], p. 45–54) we can observe that

$$
\begin{aligned}
U_{1,1}(n) &= \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(\frac{n}{2}) + \frac{1}{3}\sigma_3(\frac{n}{4}) + \frac{(2-3n)}{48}\sigma_1(n) + \frac{(1-6n)}{24}\sigma_1(\frac{n}{4}),\\
U_{1,3}(n) &= \frac{1}{192}\sigma_5(n) + \frac{1}{64}\sigma_5(\frac{n}{2}) + \frac{1}{15}\sigma_5(\frac{n}{4}) + \frac{(4-3n)}{96}\sigma_3(n) - \frac{1}{240}\sigma_1(\frac{n}{4}) - \frac{1}{64}a(n),\\
U_{3,1}(n) &= \frac{1}{3840}\sigma_5(n) + \frac{1}{256}\sigma_5(\frac{n}{2}) + \frac{1}{12}\sigma_5(\frac{n}{4}) + \frac{(1-3n)}{24}\sigma_3(\frac{n}{4}) - \frac{1}{240}\sigma_1(n)\\
&\quad + \frac{1}{256}a(n),\\
U_{1,5}(n) &= \frac{1}{408}\sigma_7(n) + \frac{1}{136}\sigma_7(\frac{n}{2}) + \frac{32}{1071}\sigma_7(\frac{n}{4}) + \frac{(2-n)}{48}\sigma_5(n) + \frac{1}{504}\sigma_1(\frac{n}{4})\\
&\quad - \frac{19}{816}b(n) - \frac{26}{51}b(\frac{n}{2}),\\
U_{3,3}(n) &= \frac{1}{32640}\sigma_7(n) + \frac{1}{2176}\sigma_7(\frac{n}{2}) + \frac{2}{255}\sigma_7(\frac{n}{4}) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{4})\\
&\quad + \frac{9}{2176}b(n) + \frac{9}{136}b(\frac{n}{2}),\\
U_{5,1}(n) &= \frac{1}{137088}\sigma_7(n) + \frac{1}{2176}\sigma_7(\frac{n}{2}) + \frac{2}{51}\sigma_7(\frac{n}{4}) + \frac{(1-2n)}{24}\sigma_5(\frac{n}{4}) + \frac{1}{504}\sigma_1(n)\\
&\quad - \frac{13}{6528}b(n) - \frac{19}{816}b(\frac{n}{2}),\\
U_{1,9}(n) &= -\frac{7}{16584}\sigma_{11}(n) + \frac{23}{5528}\sigma_{11}(\frac{n}{2}) + \frac{256}{22803}\sigma_{11}(\frac{n}{4}) + \frac{(10-3n)}{240}\sigma_9(n)\\
&\quad + \frac{1}{264}\sigma_1(\frac{n}{4}) - \frac{1589}{55280}\tau(n) - \frac{5790}{691}\tau(\frac{n}{2}) + \frac{2562304}{691}\tau(\frac{n}{4}) + 61440f(n),\\
U_{3,7}(n) &= \frac{121}{2653440}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{8}{10365}\sigma_{11}(\frac{n}{4}) - \frac{1}{240}\sigma_7(n)\\
&\quad - \frac{1}{480}\sigma_3(\frac{n}{4}) + \frac{729}{176896}\tau(n) + \frac{6003}{11056}\tau(\frac{n}{2}) - \frac{71496}{691}\tau(\frac{n}{4}) - 1920f(n),\\
U_{5,5}(n) &= \frac{1}{11144448}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{16}{43533}\sigma_{11}(\frac{n}{4}) + \frac{1}{504}\sigma_5(n)\\
&\quad + \frac{1}{504}\sigma_5(\frac{n}{4}) - \frac{351}{176896}\tau(n) - \frac{2505}{22112}\tau(\frac{n}{2}) - \frac{5616}{691}\tau(\frac{n}{4}),\\
U_{9,1}(n) &= \frac{31}{5837568}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{31}{2073}\sigma_{11}(\frac{n}{4}) + \frac{(5-6n)}{120}\sigma_9(\frac{n}{4})\\
&\quad + \frac{1}{264}\sigma_1(n) - \frac{671}{176896}\tau(n) - \frac{2505}{22112}\tau(\frac{n}{2}) - \frac{105314}{3455}\tau(\frac{n}{4}) - 240f(n)
\end{aligned}
$$

and in ([9], (21), (25)) we can see that

$$
\begin{aligned}
U_{1,7}(n) &= \frac{17}{11904}\sigma_9(n) + \frac{17}{3968}\sigma_9(\frac{n}{2}) + \frac{8}{465}\sigma_9(\frac{n}{4}) + \frac{(8-3n)}{192}\sigma_7(n) - \frac{1}{480}\sigma_1(\frac{n}{4}) \\
&\quad + \frac{433}{248}d(n) - \frac{4232}{31}d(\frac{n}{2}) - \frac{109}{3968}c(n) - \frac{529}{124}c(\frac{n}{2}), \\
U_{3,5}(n) &= \frac{1}{119040}\sigma_9(n) + \frac{1}{7936}\sigma_9(\frac{n}{2}) + \frac{4}{1953}\sigma_9(\frac{n}{4}) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(\frac{n}{4}) \\
&\quad - \frac{27}{496}d(n) + \frac{252}{31}d(\frac{n}{2}) + \frac{33}{7936}c(n) + \frac{63}{248}c(\frac{n}{2}), \\
U_{5,3}(n) &= \frac{1}{1999872}\sigma_9(n) + \frac{1}{31744}\sigma_9(\frac{n}{2}) + \frac{1}{465}\sigma_9(\frac{n}{4}) - \frac{1}{240}\sigma_5(\frac{n}{4}) + \frac{1}{504}\sigma_3(n) \\
&\quad - \frac{33}{1984}d(n) - \frac{33}{31}d(\frac{n}{2}) - \frac{63}{31744}c(n) - \frac{33}{992}c(\frac{n}{2}), \\
U_{7,1}(n) &= \frac{1}{3809280}\sigma_9(n) + \frac{17}{253952}\sigma_9(\frac{n}{2}) + \frac{17}{744}\sigma_9(\frac{n}{4}) + \frac{(2-3n)}{48}\sigma_7(\frac{n}{4}) \\
&\quad - \frac{1}{480}\sigma_1(n) + \frac{407}{15872}d(n) + \frac{109}{248}d(\frac{n}{2}) + \frac{529}{253952}c(n) + \frac{109}{7936}c(\frac{n}{2}), \\
U_{7,3}(n) &= -\frac{7}{2653440}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{17}{20730}\sigma_{11}(\frac{n}{4}) - \frac{1}{240}\sigma_7(\frac{n}{4}) \\
&\quad - \frac{1}{480}\sigma_3(n) + \frac{369}{176896}\tau(n) + \frac{1641}{22112}\tau(\frac{n}{2}) + \frac{22203}{1382}\tau(\frac{n}{4}) + 120f(n).
\end{aligned}
\tag{1.16}
$$

In this paper, we attempt to find the convolution sum formulae of

$$\sum_{m<\frac{n}{4}} m\sigma_3(m)\sigma_5(n-4m) \quad \text{and} \quad \sum_{m<\frac{n}{4}} m\sigma_5(m)\sigma_3(n-4m)$$

by trying to reduce the number of the coefficients and so we obtain the following theorem :

Theorem 1.2. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$
\begin{aligned}
&U_{m,3,5}(n) = \sum_{m<\frac{n}{4}} m\sigma_3(m)\sigma_5(n-4m) \\
&= \begin{cases}
-\frac{1}{17273894400}\Big\{19530\sigma_{11}(n) - 19530\sigma_{11}(\frac{n}{2}) - 14511n\sigma_9(n) \\
\quad -217665n\sigma_9(\frac{n}{2}) - 3537920n\sigma_9(\frac{n}{4}) - 8568400n\sigma_3(\frac{n}{4}) \\
\quad +7177926\tau(n) + 607304880\tau(\frac{n}{2}) - 26322124800\tau(\frac{n}{4}) \\
\quad -884423393280f(n) - 7182945nd(2n) - 114927120nd(n)\Big\}, & \text{for even } n, \\
-\frac{1}{822566400}\{930\sigma_{11}(n) - 691n\sigma_9(n) + 341806\tau(n) \\
\quad -42115399680f(n) - 342045nd(2n) + 15423120nd(n)\}, & \text{for odd } n,
\end{cases}
\end{aligned}
$$

(b)

$$U_{m,5,3}(n) = \sum_{m<\frac{n}{4}} m\sigma_5(m)\sigma_3(n-4m)$$
$$= \begin{cases} -\dfrac{1}{46063718400}\Big\{13020\sigma_{11}(n) - 13020\sigma_{11}(\dfrac{n}{2}) - 3455n\sigma_9(n) \\ \quad -217665n\sigma_9(\dfrac{n}{2}) - 14859264n\sigma_9(\dfrac{n}{4}) + 47983040n\sigma_5(\dfrac{n}{4}) \\ \quad -13722460\tau(n) - 602773920\tau(\dfrac{n}{2}) - 93355802624\tau(\dfrac{n}{4}) \\ \quad -589615595520f(n) + 13712895nd(2n) - 94031280nd(n)\Big\}, & \textit{for even } n, \\ -\dfrac{1}{9212743680}\{2604\sigma_{11}(n) - 691n\sigma_9(n) - 2744492\tau(n) \\ \quad -117923119104f(n) + 2742579nd(2n) - 64777104nd(n)\}, & \textit{for odd } n. \end{cases}$$

In [9] we can see more convolution sum formulae related to $U_{m,e,f}(n)$. In a similar manner to Theorem 1.2 we represent $U_{1,7}(n)$, $U_{3,5}(n)$, $U_{5,3}(n)$, and $U_{7,1}(n)$ in Corollary 2.1 without using the coefficient $c(n)$. Also we obtain Theorem 1.3 deduced from $U_{m,3,5}(n)$, $U_{m,5,3}(n)$, and etc.

Theorem 1.3. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$\begin{aligned} &L(q^4)M(q^4)N(q) \\ &= 1 + \frac{1197}{2764}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{2835}{2764}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} + \frac{64512}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{4n} \\ &\quad - \frac{189}{620}\sum_{n=1}^{\infty}n\sigma_9(n)q^n - \frac{567}{62}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} - \frac{9216}{31}\sum_{n=1}^{\infty}n\sigma_9(n)q^{4n} \\ &\quad - \frac{4881681}{13820}\sum_{n=1}^{\infty}\tau(n)q^n - \frac{11068974}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n} - \frac{1808649216}{691}\sum_{n=1}^{\infty}\tau(n)q^{4n} \\ &\quad - 18579456\sum_{n=1}^{\infty}f(n)q^n + \frac{61236}{31}\sum_{n=1}^{\infty}nd(n)q^n - \frac{18289152}{31}\sum_{n=1}^{\infty}nd(n)q^{2n} \\ &\quad - \frac{18711}{124}\sum_{n=1}^{\infty}nc(n)q^n - \frac{571536}{31}\sum_{n=1}^{\infty}nc(n)q^{2n}, \end{aligned}$$

(b)

$$
\begin{aligned}
&L(q)M(q^4)N(q)\\
&= 1 + \frac{2874}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{1206}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} + \frac{61440}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{4n}\\
&\quad - \frac{189}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^n - \frac{1134}{31}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} - \frac{36864}{31}\sum_{n=1}^{\infty}n\sigma_9(n)q^{4n}\\
&\quad + \frac{250974}{3455}\sum_{n=1}^{\infty}\tau(n)q^n + \frac{19713024}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n} - \frac{7217418240}{691}\sum_{n=1}^{\infty}\tau(n)q^{4n}\\
&\quad - 171638784\sum_{n=1}^{\infty}f(n)q^n + \frac{244944}{31}\sum_{n=1}^{\infty}nd(n)q^n - \frac{73156608}{31}\sum_{n=1}^{\infty}nd(n)q^{2n}\\
&\quad - \frac{18711}{31}\sum_{n=1}^{\infty}nc(n)q^n - \frac{2286144}{31}\sum_{n=1}^{\infty}nc(n)q^{2n},
\end{aligned}
$$

(c)

$$
\begin{aligned}
&L(q^2)M(q)N(q^4)\\
&= 1 - \frac{138}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{1146}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} + \frac{64512}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{4n}\\
&\quad - \frac{9}{248}\sum_{n=1}^{\infty}n\sigma_9(n)q^n - \frac{567}{124}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} - \frac{96768}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^{4n}\\
&\quad + \frac{66474}{691}\sum_{n=1}^{\infty}\tau(n)q^n + \frac{1561896}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n} + \frac{3298434048}{3455}\sum_{n=1}^{\infty}\tau(n)q^{4n}\\
&\quad + 9289728\sum_{n=1}^{\infty}f(n)q^n + \frac{37422}{31}\sum_{n=1}^{\infty}nd(n)q^n + \frac{4790016}{31}\sum_{n=1}^{\infty}nd(n)q^{2n}\\
&\quad + \frac{35721}{248}\sum_{n=1}^{\infty}nc(n)q^n + \frac{149688}{31}\sum_{n=1}^{\infty}nc(n)q^{2n},
\end{aligned}
$$

(d)

$$
\begin{aligned}
&L(q^4)M(q)N(q^4)\\
&= 1 - \frac{651}{2764}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{1611}{2764}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} + \frac{65280}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{4n}\\
&\quad - \frac{9}{496}\sum_{n=1}^{\infty}n\sigma_9(n)q^n - \frac{567}{248}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} - \frac{48384}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^{4n}\\
&\quad + \frac{465003}{2764}\sum_{n=1}^{\infty}\tau(n)q^n + \frac{3720114}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n} + \frac{4700507904}{3455}\sum_{n=1}^{\infty}\tau(n)q^{4n}\\
&\quad + 10727424\sum_{n=1}^{\infty}f(n)q^n + \frac{18711}{31}\sum_{n=1}^{\infty}nd(n)q^n + \frac{2395008}{31}\sum_{n=1}^{\infty}nd(n)q^{2n}\\
&\quad + \frac{35721}{496}\sum_{n=1}^{\infty}nc(n)q^n + \frac{74844}{31}\sum_{n=1}^{\infty}nc(n)q^{2n}.
\end{aligned}
$$

2 Proof of Theorem 1.1

We introduce Proposition 2.1 to obtain some convolution sum formulae, for example, $U_{m,3,5}(n)$, $M(q^2)B(q)$, and etc.

Proposition 2.1. *(See [6]) For* $q \in \mathbb{C}$ *with* $|q| < 1$*, we have*

(a)

$$M(q)M^2(q^2) = \frac{11}{78}M^3(q) + \frac{196}{39}M^3(q^2) - \frac{25}{182}N^2(q) - \frac{1100}{273}N^2(q^2),$$

(b)

$$M^2(q)M(q^2) = \frac{49}{156}M^3(q) + \frac{1408}{39}M^3(q^2) - \frac{275}{1092}N^2(q) - \frac{3200}{91}N^2(q^2),$$

(c)

$$N(q)N(q^2) = -\frac{147}{520}M^3(q) - \frac{1176}{65}M^3(q^2) + \frac{31}{104}N^2(q) + \frac{248}{13}N^2(q^2),$$

(d)

$$N(q)N(q^4) = -\frac{4851}{16640}M^3(q) - \frac{69237}{4160}M^3(q^2) - \frac{77616}{65}M^3(q^4) + \frac{971}{3328}N^2(q) + \frac{3465}{208}N^2(q^2) + \frac{15536}{13}N^2(q^4),$$

(e)

$$L(q)M(q^4) = 4L(q^4)M(q^4) + \frac{1}{336}N(q) + \frac{5}{112}N(q^2) - \frac{64}{21}N(q^4) - \frac{45}{2}A(q),$$

(f)

$$L(q)N(q^2) = 2L(q^2)N(q^2) + \frac{1}{85}M^2(q) - \frac{86}{85}M^2(q^2) - \frac{504}{17}B(q),$$

(g)

$$N(q)L(q^2) = \frac{1}{2}L(q)N(q) - \frac{43}{170}M^2(q) + \frac{64}{85}M^2(q^2) - \frac{2016}{17}B(q),$$

(h)

$$L(q^4)N(q) = \frac{1}{4}L(q)N(q) - \frac{16}{85}M^2(q) + \frac{63}{340}M^2(q^2) + \frac{64}{85}M^2(q^4) - \frac{4788}{17}B(q) - \frac{104832}{17}B(q^2),$$

(i)

$$M(q)M(q^4) = \frac{1}{272}M^2(q) + \frac{15}{272}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{4050}{17}B(q) + \frac{64800}{17}B(q^2),$$

(j)

$$M(q)M^2(q^4) = \frac{57}{416}M^3(q) + \frac{515}{104}M^3(q^2) + \frac{13932}{13}M^3(q^4) - \frac{175}{1248}N^2(q) - \frac{450}{91}N^2(q^2) - \frac{292300}{273}N^2(q^4) + 13824000F(q),$$

(k)

$$\begin{aligned}M^2(q)M(q^4) &= \frac{129}{416}M^3(q) + \frac{3765}{104}M^3(q^2) - \frac{89664}{13}M^3(q^4) - \frac{2225}{8736}N^2(q)\\ &- \frac{13175}{364}N^2(q^2) + \frac{1883200}{273}N^2(q^4) - 221184000F(q),\end{aligned}$$

(l)

$$\begin{aligned}L(q^4)M(q)N(q) &= \frac{1}{4}L(q)M(q)N(q) - \frac{853}{3120}M^3(q) - \frac{3971}{130}M^3(q^2)\\ &+ \frac{2651392}{195}M^3(q^4) - \frac{1}{208}N^2(q) + \frac{2805}{91}N^2(q^2)\\ &- \frac{1237248}{91}N^2(q^4) + 389283840F(q).\end{aligned}$$

Proof of Theorem 1.1. Proof begins by ([9],Theorem 1.1)

$$e(n) = -\frac{1}{8}\left\{d(n) - 32d(\frac{n}{2}) - c(\frac{n}{2})\right\} \quad \text{for} \quad n \in \mathbb{N}, \tag{2.1}$$

where $\sum_{n=1}^{\infty} e(n)q^n = \left(\Delta(q^2)\Delta(q^4)^4\right)^{\frac{1}{6}} = q^3\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^{16}$.
Then we can know that $e(2N) = 0$ for $N \in \mathbb{N}$, which constructs Eq. (2.1) as

$$e(2N) = 0 = -\frac{1}{8}\left\{d(2N) - 32d(\frac{2N}{2}) - c(\frac{2N}{2})\right\}$$

and so

$$d(2N) - 32d(\frac{2N}{2}) - c(\frac{2N}{2}) = 0.$$

Therefore we obtain

$$c(N) = d(2N) - 32d(N) \quad \text{for} \quad N \in \mathbb{N}.$$

□

Remark 2.1. Because of the property of Theorem 1.1, we can delete the coefficient $c(n)$ and so the convolution sums formulae related to $c(n)$ are simplified more :

$$\begin{aligned}T_{1,7}(n) &= \frac{17}{2976}\sigma_9(n) + \frac{8}{465}\sigma_9(\frac{n}{2}) + \frac{(4-3n)}{96}\sigma_7(n) - \frac{1}{480}\sigma_1(\frac{n}{2}) - \frac{1}{62}d(2n),\\ T_{3,5}(n) &= \frac{1}{7440}\sigma_9(n) + \frac{4}{1953}\sigma_9(\frac{n}{2}) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(\frac{n}{2}) + \frac{1}{248}d(2n),\\ T_{5,3}(n) &= \frac{1}{31248}\sigma_9(n) + \frac{1}{465}\sigma_9(\frac{n}{2}) - \frac{1}{240}\sigma_5(\frac{n}{2}) + \frac{1}{504}\sigma_3(n) - \frac{1}{496}d(2n),\\ T_{7,1}(n) &= \frac{1}{14880}\sigma_9(n) + \frac{17}{744}\sigma_9(\frac{n}{2}) + \frac{(2-3n)}{48}\sigma_7(\frac{n}{2}) - \frac{1}{480}\sigma_1(n) + \frac{1}{496}d(2n)\end{aligned} \tag{2.2}$$

similarly, Proposition 1.1 (b) and (c) can be rewritten as

$$\begin{aligned}
T_{m,1,5}(n) &= \frac{1}{17136}\left\{21n\sigma_7(n) + 64n\sigma_7(\frac{n}{2}) - 51n^2\sigma_5(n) + 17n\sigma_1(\frac{n}{2}) + 51d(2n)\right.\\
&\quad \left. -21nb(n)\right\},\\
T_{m,3,3}(n) &= \frac{1}{16320}\left\{2n\sigma_7(n) + 32n\sigma_7(\frac{n}{2}) - 34n\sigma_3(\frac{n}{2}) - 17d(2n) + 15nb(n)\right\},\\
T_{m,5,1}(n) &= \frac{1}{22848}\left\{4n\sigma_7(n) + 336n\sigma_7(\frac{n}{2}) - 816n^2\sigma_5(\frac{n}{2}) + 476n\sigma_5(\frac{n}{2}) + 17d(2n)\right.\\
&\quad \left. -21nb(n)\right\},\\
T_{m,1,7}(n) &= \frac{1}{446400}\left\{255n\sigma_9(n) + 768n\sigma_9(\frac{n}{2}) - 775n^2\sigma_7(n) - 465n\sigma_1(\frac{n}{2})\right.\\
&\quad \left. +1240\tau(n) + 190464\tau(\frac{n}{2}) - 720nd(2n)\right\}, \qquad (2.3)\\
T_{m,3,5}(n) &= \frac{1}{781200}\left\{21n\sigma_9(n) + 320n\sigma_9(\frac{n}{2}) + 775n\sigma_3(\frac{n}{2}) - 651\tau(n) - 59520\tau(\frac{n}{2})\right.\\
&\quad \left. +630nd(2n)\right\},\\
T_{m,5,3}(n) &= \frac{1}{520800}\left\{5n\sigma_9(n) + 336n\sigma_9(\frac{n}{2}) - 1085n\sigma_5(\frac{n}{2}) + 310\tau(n) + 13888\tau(\frac{n}{2})\right.\\
&\quad \left. -315nd(2n)\right\},\\
T_{m,7,1}(n) &= \frac{1}{111600}\left\{3n\sigma_9(n) + 1020n\sigma_9(\frac{n}{2}) - 775n\,(4n-3)\,\sigma_7(\frac{n}{2}) - 93\tau(n)\right.\\
&\quad \left. -2480\tau(\frac{n}{2}) + 90nd(2n)\right\}.
\end{aligned}$$

As the same aim of Remark 2.1, we consider the following corollary.

Corollary 2.1. *Let $n \in \mathbb{N}$. Then we have*

$$U_{1,7}(n) = \begin{cases} \frac{17}{11904}\sigma_9(n) + \frac{17}{3968}\sigma_9(\frac{n}{2}) + \frac{8}{465}\sigma_9(\frac{n}{4}) + \frac{(8-3n)}{192}\sigma_7(n) \\ \quad -\frac{1}{480}\sigma_1(\frac{n}{4}) - \frac{109}{3968}d(2n) - \frac{407}{248}d(n), & \textit{for even } n, \\ \frac{17}{11904}\sigma_9(n) + \frac{(8-3n)}{192}\sigma_7(n) - \frac{109}{3968}d(2n) + \frac{21}{8}d(n), & \textit{for odd } n, \end{cases}$$

$$U_{3,5}(n) = \begin{cases} \frac{1}{119040}\sigma_9(n) + \frac{1}{7936}\sigma_9(\frac{n}{2}) + \frac{4}{1953}\sigma_9(\frac{n}{4}) - \frac{1}{240}\sigma_5(n) \\ \quad +\frac{1}{504}\sigma_3(\frac{n}{4}) + \frac{33}{7936}d(2n) + \frac{33}{496}d(n), & \textit{for even } n, \\ \frac{1}{119040}\sigma_9(n) - \frac{1}{240}\sigma_5(n) + \frac{33}{7936}d(2n) - \frac{3}{16}d(n), & \textit{for odd } n, \end{cases}$$

$$U_{5,3}(n) = \begin{cases} \frac{1}{1999872}\sigma_9(n) + \frac{1}{31744}\sigma_9(\frac{n}{2}) + \frac{1}{465}\sigma_9(\frac{n}{4}) - \frac{1}{240}\sigma_5(\frac{n}{4}) \\ \quad +\frac{1}{504}\sigma_3(n) - \frac{63}{31744}d(2n) + \frac{27}{1984}d(n), & \textit{for even } n, \\ \frac{1}{1999872}\sigma_9(n) + \frac{1}{504}\sigma_3(n) - \frac{63}{31744}d(2n) + \frac{3}{64}d(n), & \textit{for odd } n, \end{cases}$$

$$U_{7,1}(n)=\begin{cases}\frac{1}{3809280}\sigma_9(n)+\frac{17}{253952}\sigma_9(\frac{n}{2})+\frac{17}{744}\sigma_9(\frac{n}{4})\\ \quad+\frac{(2-3n)}{48}\sigma_7(\frac{n}{4})-\frac{1}{480}\sigma_1(n)+\frac{529}{253952}d(2n)\\ \quad-\frac{433}{15872}d(n), & \textit{for even } n,\\ \frac{1}{3809280}\sigma_9(n)-\frac{1}{480}\sigma_1(n)+\frac{529}{253952}d(2n)-\frac{21}{512}d(n), & \textit{for odd } n.\end{cases}$$

Proof. Since proofs are similar, so we only prove the case of $U_{1,7}(n)$. First if n is even, i.e., $n = 2l$ with $l \in \mathbb{N}$ then we have

$$c(\frac{n}{2})=c(\frac{2l}{2})=c(l)=d(2l)-32d(l)=d(n)-32d(\frac{n}{2}), \tag{2.4}$$

where we use Theorem 1.1. And if n is odd then it is obvious that $c(\frac{n}{2}) = 0$. Now, applying Theorem 1.1 to $U_{1,7}(n)$ in (1.16), we have

$$\begin{aligned}U_{1,7}(n)=&\frac{17}{11904}\sigma_9(n)+\frac{17}{3968}\sigma_9(\frac{n}{2})+\frac{8}{465}\sigma_9(\frac{n}{4})+\frac{(8-3n)}{192}\sigma_7(n)-\frac{1}{480}\sigma_1(\frac{n}{4})\\ &-\frac{109}{3968}d(2n)+\frac{21}{8}d(n)-\frac{4232}{31}d(\frac{n}{2})-\frac{529}{124}c(\frac{n}{2}).\end{aligned} \tag{2.5}$$

Therefore if n is even then by (2.4), we can write (2.5) as

$$\begin{aligned}U_{1,7}(n)=&\frac{17}{11904}\sigma_9(n)+\frac{17}{3968}\sigma_9(\frac{n}{2})+\frac{8}{465}\sigma_9(\frac{n}{4})+\frac{(8-3n)}{192}\sigma_7(n)-\frac{1}{480}\sigma_1(\frac{n}{4})\\ &-\frac{109}{3968}d(2n)-\frac{407}{248}d(n).\end{aligned}$$

Also if n is odd then (2.5) is simplified as

$$U_{1,7}(n)=\frac{17}{11904}\sigma_9(n)+\frac{(8-3n)}{192}\sigma_7(n)-\frac{109}{3968}d(2n)+\frac{21}{8}d(n).$$

□

3 Proof of Theorem 1.2

In ([7], Theorem 1.1) we can see that

$$L(q^2)M(q)=1+360\sum_{n=1}^{\infty}n\sigma_3(n)q^n-120\sum_{n=1}^{\infty}\sigma_5(n)q^n-384\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}. \tag{3.1}$$

Similarly, we can obtain the following corollary :

Corollary 3.1. *For $q \in \mathbb{C}$ with $|q| < 1$, we obtain*

$$\begin{aligned}
&L(q^2)M(q^2)N(q)\\
&= 1 + \frac{1008}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{64512}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} - \frac{1512}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^n\\
&\quad - \frac{9216}{31}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} - \frac{701568}{3455}\sum_{n=1}^{\infty}\tau(n)q^n - \frac{3248640}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n}\\
&\quad - \frac{290304}{31}\sum_{n=1}^{\infty}nd(n)q^n - \frac{9072}{31}\sum_{n=1}^{\infty}nc(n)q^n.
\end{aligned}$$

Proof. By (1.4) and (1.7), we note that

$$\begin{aligned}
&L(q^2)M(q^2)N(q) = N(q)\cdot L(q^2)M(q^2)\\
&= \left(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)\left(1 + 720\sum_{m=1}^{\infty}m\sigma_3(m)q^{2m} - 504\sum_{m=1}^{\infty}\sigma_5(m)q^{2m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\Bigg\{360N\sigma_3(\frac{N}{2}) - 504\sigma_5(\frac{N}{2}) - 504\sigma_5(N)\\
&\quad -504\cdot 720\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\cdot m\sigma_3(m) + 504\cdot 504\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\sigma_5(m)\Bigg\}q^N \qquad (3.2)\\
&= 1 + \sum_{N=1}^{\infty}\Big\{360N\sigma_3(\frac{N}{2}) - 504\sigma_5(\frac{N}{2}) - 504\sigma_5(N) - 504\cdot 720\cdot T_{m,3,5}(N)\\
&\quad +504\cdot 504\cdot T_{5,5}(N)\Big\}q^N.
\end{aligned}$$

So we use Proposition 1.1 (c) for $T_{m,3,5}(N)$.

□

To show Lemma 3.2 (a), we need Eq. (3.3) :

$$\begin{aligned}
&L(q)M(q^2)N(q)\\
&= 1 + \frac{4080}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{61440}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} - \frac{3024}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^n\\
&\quad - \frac{18432}{31}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} + \frac{244944}{3455}\sum_{n=1}^{\infty}\tau(n)q^n + \frac{16920576}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n}\\
&\quad - \frac{580608}{31}\sum_{n=1}^{\infty}nd(n)q^n - \frac{18144}{31}\sum_{n=1}^{\infty}nc(n)q^n
\end{aligned}\qquad (3.3)$$

in ([8],Theorem 1.2 (f)). Now we can consider that Lemma 3.2 is an extension of

$$M(q)B(q) = \sum_{n=1}^{\infty}\tau(n)q^n + 256\sum_{n=1}^{\infty}\tau(n)q^{2n} \qquad (3.4)$$

in ([8], Theorem 1.2 (l)).

Lemma 3.2. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$M(q^2)B(q) = \sum_{n=1}^{\infty} \tau(n)q^n + 16\sum_{n=1}^{\infty} \tau(n)q^{2n},$$

(b)

$$\begin{aligned} M(q^4)B(q) = &-\frac{15}{11056}\sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{15}{11056}\sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} + \frac{11071}{11056}\sum_{n=1}^{\infty} \tau(n)q^n \\ &+ \frac{25997}{1382}\sum_{n=1}^{\infty} \tau(n)q^{2n} + 7680\sum_{n=1}^{\infty} \tau(n)q^{4n} + 61440\sum_{n=1}^{\infty} f(n)q^n, \end{aligned}$$

(c)

$$\begin{aligned} M(q)B(q^2) = &\frac{15}{11056}\sum_{n=1}^{\infty} \sigma_{11}(n)q^n - \frac{15}{11056}\sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} - \frac{15}{11056}\sum_{n=1}^{\infty} \tau(n)q^n \\ &- \frac{2503}{1382}\sum_{n=1}^{\infty} \tau(n)q^{2n} - 3584\sum_{n=1}^{\infty} \tau(n)q^{4n} - 61440\sum_{n=1}^{\infty} f(n)q^n. \end{aligned}$$

Proof. (a) By Proposition 2.1 (g), Corollary 3.1 can be rewritten as

$$\begin{aligned} &L(q^2)M(q^2)N(q) = L(q^2)N(q) \cdot M(q^2) \\ &= \left(\frac{1}{2}L(q)N(q) - \frac{43}{170}M^2(q) + \frac{64}{85}M^2(q^2) - \frac{2016}{17}B(q)\right)M(q^2) \\ &= \frac{1}{2}L(q)N(q)M(q^2) - \frac{43}{170}M^2(q)M(q^2) + \frac{64}{85}M^3(q^2) - \frac{2016}{17}B(q)M(q^2), \end{aligned}$$

thus we refer to (1.6), Proposition 2.1 (b), and (3.3).

(b) By Proposition 2.1 (i), we can rewrite Proposition 2.1 (j) as

$$\begin{aligned} &M(q)M^2(q^4) = M(q)M(q^4) \cdot M(q^4) \\ &= \left(\frac{1}{272}M^2(q) + \frac{15}{272}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{4050}{17}B(q) + \frac{64800}{17}B(q^2)\right)M(q^4) \\ &= \frac{1}{272}M^2(q)M(q^4) + \frac{15}{272}M^2(q^2)M(q^4) + \frac{16}{17}M^3(q^4) + \frac{4050}{17}B(q)M(q^4) \\ &\quad + \frac{64800}{17}B(q^2)M(q^4), \end{aligned}$$

so we use (1.6), Proposition 2.1 (k), and Lemma 3.2 (a).

(c) We note that by Proposition 2.1 (h)

$$\begin{aligned} &L(q^4)M(q)N(q) = L(q^4)N(q) \cdot M(q) \\ &= \left(\frac{1}{4}L(q)N(q) - \frac{16}{85}M^2(q) + \frac{63}{340}M^2(q^2) + \frac{64}{85}M^2(q^4) - \frac{4788}{17}B(q)\right. \\ &\qquad \left. - \frac{104832}{17}B(q^2)\right)M(q) \\ &= \frac{1}{4}L(q)N(q)M(q) - \frac{16}{85}M^3(q) + \frac{63}{340}M^2(q^2)M(q) + \frac{64}{85}M^2(q^4)M(q) \\ &\qquad - \frac{4788}{17}B(q)M(q) - \frac{104832}{17}B(q^2)M(q). \end{aligned}$$

Therefore we refer to (1.6), (1.10), Proposition 2.1 (a), (j), (l) and (3.4).

□

Remark 3.1. By (1.1), (1.3), and Lemma 3.2, we deduce that

$$\begin{aligned}240\sum_{N=1}^{\infty}\left(\sum_{m<\frac{N}{2}}\sigma_3(m)b(N-2m)\right)q^N &= 240\left(\sum_{n=1}^{\infty}b(n)q^n\right)\left(\sum_{m=1}^{\infty}\sigma_3(m)q^{2m}\right)\\ &= B(q)\left(M(q^2)-1\right)\\ &= B(q)M(q^2)-B(q)\end{aligned}$$

and so we conclude that

$$\sum_{m<\frac{n}{2}}\sigma_3(m)b(n-2m)=\frac{1}{240}\left\{\tau(n)+16\tau(\frac{n}{2})-b(n)\right\} \tag{3.5}$$

for $n\in\mathbb{N}$. Similarly, we obtain

$$\begin{aligned}\sum_{m<\frac{n}{4}}\sigma_3(m)b(n-4m) = -\frac{1}{2653440}&\left\{15\sigma_{11}(n)-15\sigma_{11}(\frac{n}{2})-11071\tau(n)\right.\\ &-207976\tau(\frac{n}{2})-84910080\tau(\frac{n}{4})-679280640f(n)\\ &\left.+11056b(n)\right\}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\sum_{m<\frac{n}{2}}b(m)\sigma_3(n-2m)=\frac{1}{2653440}&\left\{15\sigma_{11}(n)-15\sigma_{11}(\frac{n}{2})-15\tau(n)-20024\tau(\frac{n}{2})\right.\\ &\left.-39624704\tau(\frac{n}{4})-679280640f(n)-11056b(\frac{n}{2})\right\}.\end{aligned} \tag{3.7}$$

Proof of Theorem 1.2. (a) By (1.4) and (1.7), we obtain

$$
\begin{aligned}
&L(q^4)M(q^4)N(q) = N(q)\cdot L(q^4)M(q^4)\\
&= \left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)\left(1+720\sum_{m=1}^{\infty}m\sigma_3(m)q^{4m}-504\sum_{m=1}^{\infty}\sigma_5(m)q^{4m}\right)\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{180N\sigma_3(\frac{N}{4})-504\sigma_5(\frac{N}{4})-504\sigma_5(N)\\
&\qquad -504\cdot 720\sum_{m<\frac{N}{4}}\sigma_5(N-4m)\cdot m\sigma_3(m)\\
&\qquad +504\cdot 504\sum_{m<\frac{N}{4}}\sigma_5(N-4m)\sigma_5(m)\Bigg\}q^N\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{180N\sigma_3(\frac{N}{4})-504\sigma_5(\frac{N}{4})-504\sigma_5(N)\\
&\qquad -504\cdot 720\sum_{m<\frac{N}{4}}m\sigma_3(m)\sigma_5(N-4m)+504\cdot 504\cdot U_{5,5}(N)\Bigg\}q^N.
\end{aligned}
\tag{3.8}
$$

Also by Proposition 2.1 (h), we have

$$
\begin{aligned}
&L(q^4)M(q^4)N(q) = L(q^4)N(q)\cdot M(q^4)\\
&= \Big(\frac{1}{4}L(q)N(q)-\frac{16}{85}M^2(q)+\frac{63}{340}M^2(q^2)+\frac{64}{85}M^2(q^4)-\frac{4788}{17}B(q)\\
&\qquad -\frac{104832}{17}B(q^2)\Big)M(q^4)\\
&= \frac{1}{4}L(q)N(q)M(q^4)-\frac{16}{85}M^2(q)M(q^4)+\frac{63}{340}M^2(q^2)M(q^4)+\frac{64}{85}M^3(q^4)\\
&\qquad -\frac{4788}{17}B(q)M(q^4)-\frac{104832}{17}B(q^2)M(q^4)\\
&= \frac{1}{4}L(q)N(q)M(q^4)+\Bigg[\frac{3}{4}+\sum_{N=1}^{\infty}\Bigg\{-\frac{1677}{2764}\sigma_{11}(N)+\frac{1629}{2764}\sigma_{11}(\frac{N}{2})+\frac{49152}{691}\sigma_{11}(\frac{N}{4})\\
&\qquad -\frac{1026531}{2764}\tau(N)-\frac{15997230}{691}\tau(\frac{N}{2})-\frac{4294656}{691}\tau(\frac{N}{4})+24330240f(N)\Bigg\}q^N\Bigg],
\end{aligned}
\tag{3.9}
$$

where we use (1.6), Proposition 2.1 (b), (k), Lemma 3.2 (a) and (b). Then by letting

$$
\begin{aligned}
\alpha(N) := &-\frac{1677}{2764}\sigma_{11}(N)+\frac{1629}{2764}\sigma_{11}(\frac{N}{2})+\frac{49152}{691}\sigma_{11}(\frac{N}{4})-\frac{1026531}{2764}\tau(N)\\
&-\frac{15997230}{691}\tau(\frac{N}{2})-\frac{4294656}{691}\tau(\frac{N}{4})+24330240f(N),
\end{aligned}
\tag{3.10}
$$

we can rewrite Eq. (3.9) as

$$\begin{aligned}
&L(q^4)M(q^4)N(q) \\
&= \frac{1}{4}L(q)N(q)\cdot M(q^4) + \frac{3}{4} + \sum_{N=1}^{\infty}\alpha(N)q^N \\
&= \frac{1}{4}\left(1 - 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right)\left(1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^{4m}\right) + \frac{3}{4} \\
&\quad + \sum_{N=1}^{\infty}\alpha(N)q^N \\
&= \frac{1}{4}\Bigg[1 + \sum_{N=1}^{\infty}\Big\{-1008N\sigma_5(N) + 480\sigma_7(N) + 240\sigma_3(\frac{N}{4}) \\
&\quad - 1008\cdot 240\sum_{m<\frac{N}{4}}(N-4m)\,\sigma_5(N-4m)\sigma_3(m) \\
&\quad +480\cdot 240\sum_{m<\frac{N}{4}}\sigma_7(N-4m)\sigma_3(m)\Big\}q^N\Bigg] + \frac{3}{4} + \sum_{N=1}^{\infty}\alpha(N)q^N \\
&= 1 + \sum_{N=1}^{\infty}\Big\{-252N\sigma_5(N) + 120\sigma_7(N) + 60\sigma_3(\frac{N}{4}) - 252\cdot 240N\cdot U_{3,5}(N) \\
&\quad +252\cdot 240\cdot 4\sum_{m<\frac{N}{4}} m\sigma_3(m)\sigma_5(N-4m) + 120\cdot 240\cdot U_{3,7}(N) + \alpha(N)\Big\}q^N,
\end{aligned} \tag{3.11}$$

where we use (1.3) and (1.9) for the third line. So we equate (3.8) with (3.11) and use (1.16) for $U_{3,5}(N)$ then we obtain

$$\begin{aligned}
&U_{m,3,5}(n) = \sum_{m<\frac{n}{4}} m\sigma_3(m)\sigma_5(n-4m) \\
&= -\frac{1}{17273894400}\Big\{19530\sigma_{11}(n) - 19530\sigma_{11}(\frac{n}{2}) - 14511n\sigma_9(n) - 217665n\sigma_9(\frac{n}{2}) \\
&\quad - 3537920n\sigma_9(\frac{n}{4}) - 8568400n\sigma_3(\frac{n}{4}) + 7177926\tau(n) + 607304880\tau(\frac{n}{2}) \\
&\quad - 26322124800\tau(\frac{n}{4}) - 884423393280f(n) + 94031280nd(n) \\
&\quad -14042004480nd(\frac{n}{2}) - 7182945nc(n) - 438812640nc(\frac{n}{2})\Big\}.
\end{aligned} \tag{3.12}$$

Thus applying Theorem 1.1 into (3.12) we have

$$\begin{aligned}
U_{m,3,5}(n) = -\frac{1}{17273894400}&\Big\{19530\sigma_{11}(n) - 19530\sigma_{11}(\frac{n}{2}) - 14511n\sigma_9(n) \\
&- 217665n\sigma_9(\frac{n}{2}) - 3537920n\sigma_9(\frac{n}{4}) - 8568400n\sigma_3(\frac{n}{4}) + 7177926\tau(n) \\
&+ 607304880\tau(\frac{n}{2}) - 26322124800\tau(\frac{n}{4}) - 884423393280f(n) \\
&- 7182945nd(2n) + 323885520nd(n) - 14042004480nd(\frac{n}{2}) \\
&-438812640nc(\frac{n}{2})\Big\}.
\end{aligned} \tag{3.13}$$

Now, if n is even then by (2.4), Eq. (3.13) can be written as

$$\begin{aligned} U_{m,3,5}(n) = -\frac{1}{17273894400}\Big\{&19530\sigma_{11}(n) - 19530\sigma_{11}(\frac{n}{2}) - 14511n\sigma_9(n) \\ &- 217665n\sigma_9(\frac{n}{2}) - 3537920n\sigma_9(\frac{n}{4}) - 8568400n\sigma_3(\frac{n}{4}) + 7177926\tau(n) \\ &+ 607304880\tau(\frac{n}{2}) - 26322124800\tau(\frac{n}{4}) - 884423393280f(n) \\ &-7182945nd(2n) - 114927120nd(n)\} . \end{aligned}$$

On the other hand, if n is odd then we can simplify Eq. (3.13) as

$$\begin{aligned} U_{m,3,5}(n) = -\frac{1}{17273894400}\{&19530\sigma_{11}(n) - 14511n\sigma_9(n) + 7177926\tau(n) \\ &-884423393280f(n) - 7182945nd(2n) + 323885520nd(n)\} . \end{aligned}$$

(b) By (1.4) and (3.1), we have

$$\begin{aligned} &L(q^2)M(q)N(q^4) = L(q^2)M(q) \cdot N(q^4) \\ &= \left(1 + 360\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120\sum_{n=1}^{\infty}\sigma_5(n)q^n - 384\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right) \\ &\quad \times \left(1 - 504\sum_{m=1}^{\infty}\sigma_5(m)q^{4m}\right) \\ &= 1 + \sum_{N=1}^{\infty}\Bigg\{360N\sigma_3(N) - 120\sigma_5(N) - 384\sigma_5(\frac{N}{2}) - 504\sigma_5(\frac{N}{4}) \\ &\quad - 360 \cdot 504 \sum_{m<\frac{N}{4}} (N-4m)\,\sigma_3(N-4m)\sigma_5(m) \\ &\quad +120 \cdot 504 \sum_{m<\frac{N}{4}} \sigma_5(N-4m)\sigma_5(m) + 384 \cdot 504 \sum_{m<\frac{N}{4}} \sigma_5(\frac{N}{2} - 2m)\sigma_5(m)\Bigg\} q^N \qquad (3.14) \\ &= 1 + \sum_{N=1}^{\infty}\Bigg\{360N\sigma_3(N) - 120\sigma_5(N) - 384\sigma_5(\frac{N}{2}) - 504\sigma_5(\frac{N}{4}) \\ &\quad - 360 \cdot 504N \cdot U_{5,3}(N) + 360 \cdot 504 \cdot 4\sum_{m<\frac{N}{4}} m\sigma_5(m)\sigma_3(N-4m) \\ &\quad +120 \cdot 504 \cdot U_{5,5}(N) + 384 \cdot 504 \cdot T_{5,5}(\frac{N}{2})\Bigg\} q^N. \end{aligned}$$

And by Proposition 2.1 (f) we can observe that

$$
\begin{aligned}
&L(q^2)M(q)N(q^4) = M(q)\cdot L(q^2)N(q^4)\\
&= M(q)\left(2L(q^4)N(q^4) + \frac{1}{85}M^2(q^2) - \frac{86}{85}M^2(q^4) - \frac{504}{17}B(q^2)\right)\\
&= 2M(q)L(q^4)N(q^4) + \frac{1}{85}M(q)M^2(q^2) - \frac{86}{85}M(q)M^2(q^4) - \frac{504}{17}M(q)B(q^2)\\
&= 2M(q)L(q^4)N(q^4) + \left[-1 + \sum_{N=1}^{\infty}\left\{\frac{375}{1382}\sigma_{11}(N) + \frac{681}{1382}\sigma_{11}(\frac{N}{2}) - \frac{66048}{691}\sigma_{11}(\frac{N}{4})\right.\right.\\
&\quad \left.\left.-\frac{332055}{1382}\tau(N) - \frac{5878332}{691}\tau(\frac{N}{2}) - \frac{1220516352}{691}\tau(\frac{N}{4}) - 12165120f(N)\right\}q^N\right],
\end{aligned}
\tag{3.15}
$$

where we use Proposition 2.1 (a), (j), and Lemma 3.2 (c). Then putting

$$
\begin{aligned}
\beta(N) := \frac{375}{1382}\sigma_{11}(N) + \frac{681}{1382}\sigma_{11}(\frac{N}{2}) - \frac{66048}{691}\sigma_{11}(\frac{N}{4}) - \frac{332055}{1382}\tau(N)\\
- \frac{5878332}{691}\tau(\frac{N}{2}) - \frac{1220516352}{691}\tau(\frac{N}{4}) - 12165120f(N),
\end{aligned}
\tag{3.16}
$$

constructs Eq. (3.15) as

$$
\begin{aligned}
&L(q^2)M(q)N(q^4)\\
&= 2M(q)\cdot L(q^4)N(q^4) - 1 + \sum_{N=1}^{\infty}\beta(N)q^N\\
&= 2\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\left(1 - 1008\sum_{m=1}^{\infty}m\sigma_5(m)q^{4m} + 480\sum_{m=1}^{\infty}\sigma_7(m)q^{4m}\right) - 1\\
&\quad + \sum_{N=1}^{\infty}\beta(N)q^N\\
&= 2\left[1 + \sum_{N=1}^{\infty}\left\{-252N\sigma_5(\frac{N}{4}) + 480\sigma_7(\frac{N}{4}) + 240\sigma_3(N)\right.\right.\\
&\quad - 240\cdot 1008\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\cdot m\sigma_5(m)\\
&\quad \left.\left.+240\cdot 480\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\sigma_7(m)\right\}q^N\right] - 1 + \sum_{N=1}^{\infty}\beta(N)q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-504N\sigma_5(\frac{N}{4}) + 960\sigma_7(\frac{N}{4}) + 480\sigma_3(N)\right.\\
&\quad \left.-480\cdot 1008\sum_{m<\frac{N}{4}}m\sigma_5(m)\sigma_3(N-4m) + 480\cdot 480\cdot U_{7,3}(N) + \beta(N)\right\}q^N,
\end{aligned}
\tag{3.17}
$$

where we use (1.3) and (1.9) for the third line. Therefore we equate (3.14) with (3.17) and use (1.16) for $U_{5,3}(N)$ then we obtain

$$
\begin{aligned}
&U_{m,5,3}(n) = \sum_{m<\frac{n}{4}} m\sigma_5(m)\sigma_3(n-4m)\\
&= -\frac{1}{46063718400}\Big\{13020\sigma_{11}(n) - 13020\sigma_{11}(\frac{n}{2}) - 3455n\sigma_9(n) - 217665n\sigma_9(\frac{n}{2})\\
&\quad - 14859264n\sigma_9(\frac{n}{4}) + 47983040n\sigma_5(\frac{n}{4}) - 13722460\tau(n) - 602773920\tau(\frac{n}{2})\\
&\quad - 93355802624\tau(\frac{n}{4}) - 589615595520f(n) + 114927120nd(n)\\
&\quad +7355335680nd(\frac{n}{2}) + 13712895nc(n) + 229854240nc(\frac{n}{2})\Big\}.
\end{aligned} \tag{3.18}
$$

So applying Theorem 1.1 to (3.18) we have

$$
\begin{aligned}
U_{m,5,3}(n) = -\frac{1}{46063718400}&\Big\{13020\sigma_{11}(n) - 13020\sigma_{11}(\frac{n}{2}) - 3455n\sigma_9(n)\\
&- 217665n\sigma_9(\frac{n}{2}) - 14859264n\sigma_9(\frac{n}{4}) + 47983040n\sigma_5(\frac{n}{4})\\
&- 13722460\tau(n) - 602773920\tau(\frac{n}{2}) - 93355802624\tau(\frac{n}{4})\\
&- 589615595520f(n) + 13712895nd(2n) - 323885520d(n)\\
&+7355335680nd(\frac{n}{2}) + 229854240nc(\frac{n}{2})\Big\}.
\end{aligned} \tag{3.19}
$$

Finally, we easily obtain the convolution sum formula for odd n from (3.19) but we should apply (2.4) into (3.19) for even n.

□

4 Induced identities from $U_{m,3,5}(n)$ and $U_{m,5,3}(n)$

Theorem 1.3 is obtained directly in proof of Theorem 1.2 :

Proof of Theorem 1.3. (a) Insert (3.12) into (3.8).

(b) It is obvious by Theorem 1.3 (a) and (3.9).

(c) Insert (3.18) into (3.14).

(d) It is definite by Theorem 1.3 (c) and (3.15).

□

Remark 4.1. By Proposition 2.1 (e) let us rewrite Theorem 1.3 (b) as

$$
\begin{aligned}
&L(q)M(q^4)N(q) = L(q)M(q^4)\cdot N(q)\\
&= \Big(4L(q^4)M(q^4) + \frac{1}{336}N(q) + \frac{5}{112}N(q^2) - \frac{64}{21}N(q^4) - \frac{45}{2}A(q)\Big)N(q)\\
&= 4L(q^4)M(q^4)N(q) + \frac{1}{336}N^2(q) + \frac{5}{112}N(q^2)N(q) - \frac{64}{21}N(q^4)N(q)\\
&\quad - \frac{45}{2}A(q)N(q),
\end{aligned}
$$

then this leads that

$$
\begin{aligned}
&L(q)M(q^4)N(q) - 4L(q^4)M(q^4)N(q) \\
&= \frac{1}{336}N^2(q) + \frac{5}{112}N(q^2)N(q) - \frac{64}{21}N(q^4)N(q) - \frac{45}{2}A(q)N(q) \\
&= -3 - 4\sum_{n=1}^{\infty}\alpha(n)q^n,
\end{aligned}
$$

where we can deduce the last line from (3.10) and (3.11). So using (1.8), Proposition 2.1 (c) and (d) we obtain

$$
\begin{aligned}
A(q)N(q) = &-\frac{66}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{66}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} + \frac{757}{691}\sum_{n=1}^{\infty}\tau(n)q^n \\
&- \frac{194928}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n} + 278528\sum_{n=1}^{\infty}\tau(n)q^{4n} + 4325376\sum_{n=1}^{\infty}f(n)q^n.
\end{aligned}
$$

Finally, this shows that

$$
\begin{aligned}
504\sum_{N=1}^{\infty}\left(\sum_{m=1}^{N-1}\sigma_5(m)a(N-m)\right)q^N &= 504\left(\sum_{n=1}^{\infty}a(n)q^n\right)\left(\sum_{m=1}^{\infty}\sigma_5(m)q^m\right) \\
&= A(q)\left(1-N(q)\right) \\
&= A(q) - A(q)N(q)
\end{aligned}
$$

and so

$$
\begin{aligned}
\sum_{m=1}^{n-1}\sigma_5(m)a(n-m) = \frac{1}{348264}\Big\{&66\sigma_{11}(n) - 66\sigma_{11}(\frac{n}{2}) - 757\tau(n) + 194928\tau(\frac{n}{2}) \\
&-192462848\tau(\frac{n}{4}) - 2988834816f(n) + 691a(n)\Big\}
\end{aligned}
$$

for $n \in \mathbb{N}$.

5 Conclusions

In this paper, we study the convolution sum formulae mainly as

$$
U_{m,3,5}(n) = \sum_{m<\frac{n}{4}} m\sigma_3(m)\sigma_5(n-4m)
$$

and

$$
U_{m,5,3}(n) = \sum_{m<\frac{n}{4}} m\sigma_5(m)\sigma_3(n-4m)
$$

for $n(\in \mathbb{N})$. Furthermore we can deduce some identities from the above convolution sums. Especially, we obtain the coefficient relation as

$$c(n) = d(2n) - 32d(n)$$

in Theorem 1.1.

Competing Interests

The author declares that no competing interests exist.

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Appendix

The first twenty values of $\tau(n)$ are given in the Table 1,

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	6	−6048	11	534612	16	987136
2	−24	7	−16744	12	−370944	17	−6905934
3	252	8	84480	13	−577738	18	2727432
4	−1472	9	−113643	14	401856	19	10661420
5	4830	10	−115920	15	1217160	20	−7109760

TABLE 1. $\tau(n)$ for n $(1 \leq n \leq 20)$

similarly the first twenty values of $a(n)$, $b(n)$, $d(n)$, and $f(n)$ are listed in the following tables.

n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$
1	1	6	0	11	540	16	0
2	0	7	−88	12	0	17	594
3	−12	8	0	13	−418	18	0
4	0	9	−99	14	0	19	836
5	54	10	0	15	−648	20	0

TABLE 2. $a(n)$ for n $(1 \leq n \leq 20)$

n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	6	−96	11	1092	16	4096
2	−8	7	1016	12	768	17	14706
3	12	8	−512	13	1382	18	16344
4	64	9	−2043	14	−8128	19	−39940
5	−210	10	1680	15	−2520	20	−13440

TABLE 3. $b(n)$ for n $(1 \leq n \leq 20)$

n	$d(n)$	n	$d(n)$	n	$d(n)$	n	$d(n)$
1	0	6	−156	11	−536	16	4096
2	1	7	112	12	−2496	17	−17472
3	−8	8	256	13	4384	18	4653
4	16	9	−576	14	−952	19	5848
5	32	10	870	15	336	20	13920

TABLE 4. $d(n)$ for n $(1 \leq n \leq 20)$

n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$
1	0	6	8	11	6296	16	388608
2	0	7	44	12	16384	17	756822
3	0	8	192	13	39569	18	1419200
4	0	9	694	14	89424	19	2572328
5	1	10	2208	15	191028	20	4521984

TABLE 5. $f(n)$ for n $(1 \leq n \leq 20)$

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