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The Fixed Points of Abstract Morphisms

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Abstract

In this article, with the help of three axioms (Definition 3.1), the notion of abstract morphisms is introduced (see [1,2]). It will be proven that Hausdorff topological spaces together with abstract morphisms create a category on which the functor of Čech homology is extended.

Keywords: Abstract morphism, equivalency relation, strongly admissible map, Čech homology functor, fixed point, Vietoris map.

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1 Introduction

In 1976, L. Górniewicz (see [3,4]) introduced the notion of strongly admissible multi-valued mappings and proved that the composition of strongly admissible mappings is also a strongly admissible mapping. In 1983 it was L. Górniewicz (see [1]) as well that introduced the notion of a morphism, i.e. some other version of strongly admissible mappings. Morphisms, as opposed to strongly-admissible mappings, together with Hausdorff topological spaces create a category on which a functor of Čech homology is extended. In 1994, W. Kryszewski (see [2]) introduced the notion of a morphism essentially different from the morphism in the sense of Górniewicz in regard to some important applications of their properties. In this article, with the help of three axioms, the notion of an abstract morphism was

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introduced. This notion does not only encompass already existing morphisms, but also all the other morphisms that could be created on the basis of these axioms. The examples of other essentially different morphisms were given. The fixed point theorem was proven for the abstract morphisms. In terms of fixed point theory, we recommend the following publications: [4,3] as well as [5,6,7].

2 Preliminaries

Let X and Y be Hausdorff topological spaces. Assume that for every $x \in X$ a non-empty and compact subset $\varphi(x)$ of Y is given. In such a case we say that $\varphi : X \multimap Y$ is a multi-valued mapping. For a multi-valued mapping $\varphi : X \multimap Y$ and a subset $A \subset Y$, we let:

$$\varphi^{-1}(A) = \{x \in X;\ \varphi(x) \subset A\}.$$

If for every open $U \subset Y$ the set $\varphi^{-1}(U)$ is open, then φ is called an upper semi-continuous mapping; we shall write that φ is u.s.c. Let H_* be the $\check{C}ech$ homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus, for any pair (X, A), we have $H_*(X, A) = \{H_q(X, A)\}_{q\geq 0}$, a graded vector space and, for any map $f : (X, A) \to (Y, B)$, we have the induced linear map $f_* = \{f_{*q}\}_{q\geq 0} : H_*(X, A) \to H_*(Y, B)$, where $f_{*q} : H_q(X, A) \to H_q(Y, B)$ is a linear map from the q-dimensional homology $H_q(X, A)$ of the pair (X, A) into the q-dimensional homology $H_q(Y, B)$ of the pair (Y, B). If $A = \emptyset$ then $H_*(X, A) = H_*(X)$. A space X is acyclic if:

(i) X is non-empty,

(ii) $H_q(X) = 0$ for every $q \geq 1$ and

(iii) $H_0(X) \approx \mathbb{Q}$.

A continuous and closed mapping $f : X \to Y$ is called proper if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is nonempty and compact. A proper map $p : X \to Y$ is called Vietoris provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic.

Let $u : E \to E$ be an endomorphism of an arbitrary vector space. Let us put $N(u) = \{x \in E : u^n(x) = 0 \quad for\ some\ n\}$, where u^n is the nth iterate of u and $\widetilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u} : \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{u}([x]) = [u(x)]$. We call *u* admissible provided $dim\widetilde{E} < \infty$.

Let $u = \{u_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call *u* a Leray endomorphism if

(i) all u_q are admissible,

(ii) almost all $\widetilde{E_q}$ are trivial. For such *u*, we define the (generalized) Lefschetz number $\Lambda(u)$ of u by putting

$$\Lambda(u) = \sum_q (-1)^q tr(\widetilde{u_q}),$$

where $tr(\widetilde{u_q})$ is the ordinary trace of $\widetilde{u_q}$ (comp. [4]). The following important property of the Leray endomorphism is a consequence of the well-known a formula $tr(u \circ v) = tr(v \circ u)$ for the ordinary trace.

Proposition 2.1. (see ([4])) Assume that, in the category of graded vector spaces, the following diagram commutes

$$\begin{array}{ccc} E' & \xrightarrow{\ u\ } & E'' \\ {\scriptstyle u'}\uparrow & \nwarrow^{v} & \uparrow{\scriptstyle u''} \\ E' & \xrightarrow{\ u\ } & E'' \end{array}$$

Then, if u' or u'' is a Leray endomorphism, so is the other; and, in that case,

$$\Lambda(u') = \Lambda(u'').$$

An endomorphism $u : E \to E$ of a graded vector space E is called weakly nilpotent if for every $q \geq 0$ and for every $x \in E_q$, there exists an integer n such that $u_q^n(x) = 0$. Since, for a weakly nilpotent endomorphism $u : E \to E$, we have $N(u) = E$, we get:

Proposition 2.2. (see ([4])) If $u : E \to E$ is a weakly-nilpotent endomorphism, then $\Lambda(u) = 0$.

The symbol $D(X, Y)$ will denote the set of all diagrams of the form

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y,$$

where $p : Z \to X$ denotes a Vietoris map and $q : Z \to Y$ denotes a continuous map. Each such diagram will be denoted by (p, q).

Definition 2.3. (see [4]) Let $(p_1, q_1) \in D(X, Y)$ and $(p_2, q_2) \in D(Y, T)$. The composition of the diagrams

$$X \xleftarrow{\ p_1\ } Z_1 \xrightarrow{\ q_1\ } Y \xleftarrow{\ p_2\ } Z_2 \xrightarrow{\ q_2\ } T,$$

is called the diagram $(p, q) \in D(X, T)$

$$X \xleftarrow{\ p\ } Z_1 \triangle_{q_1 p_2} Z_2 \xrightarrow{\ q\ } T,$$

$$\text{where } \ Z_1 \triangle_{q_1 p_2} Z_2 = \{(z_1, z_2) \in Z_1 \times Z_2 : \ q_1(z_1) = p_2(z_2)\},$$

$$p = p_1 \circ f_1, \ \ q = q_2 \circ f_2,$$

$$Z_1 \xleftarrow{\ f_1\ } Z_1 \triangle_{q_1 p_2} Z_2 \xrightarrow{\ f_2\ } Z_2,$$

$$f_1(z_1, z_2) = z_1 \text{ (Vietoris map)}, \ \ f_2(z_1, z_2) = z_2 \ \text{ for each } \ (z_1, z_2) \in Z.$$

It shall be written

$$(p, q) = (p_2, q_2) \circ (p_1, q_1).$$

From the Theorems ((40.5), (40.6)) ([4], p. 201, 202) it also results that in Definition 2.3 the composition of the diagrams satisfies the condition:

$$\text{for each } \ x \in X \ \ q(p^{-1}(x)) = q_2(p_2^{-1}(q_1(p_1^{-1}(x)))). \tag{2.1}$$

Let $Id : X \to X$ be an identity map.

Definition 2.4. [2] Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$.

$$(p_1, q_1) \sim_k (p_2, q_2) \ \text{ (in the sense of Kryszewski)}$$

if and only if there exist spaces Z_1, Z_2 and a homeomorphism $h : Z_1 \multimap Z_2$ such that the following diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\ p_1\ } & Z_1 & \xrightarrow{\ q_1\ } & Y \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Id} \\ X & \xleftarrow{\ p_2\ } & Z_2 & \xrightarrow{\ q_2\ } & Y \end{array}$$

is commutative, that is

$$p_2 \circ h = p_1, \;\; q_2 \circ h = q_1.$$

Definition 2.5. [4,1] Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$.

$$(p_1, q_1) \sim_g (p_2, q_2) \;\; \text{(in the sense of Górniewicz)}$$

if and only if there exist spaces Z_1, Z_2 and the continuous mapping $g : Z_1 \to Z_2, \; h : Z_2 \to Z_1$ such that the following diagrams:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \big\downarrow{\scriptstyle Id} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle Id} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y, \end{array} \qquad \begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \big\uparrow{\scriptstyle Id} & & \big\uparrow{\scriptstyle h} & & \big\uparrow{\scriptstyle Id} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y, \end{array}$$

are commutative, that is

$$p_2 \circ g = p_1, \;\; q_2 \circ g = q_1 \;\; \text{and} \;\; p_1 \circ h = p_2, \;\; q_1 \circ h = q_2.$$

Theorem 2.6. [2,4,1] The relations introduced in Definitions 2.4 and 2.5 are equivalency relations in the set $D(X, Y)$.

Recall that if $p : X \to Y$ is a Vietoris map then $p_* : H_*(X) \to H_*(Y)$ is an isomorphism. Let $(p, q) \in D(X, Y)$. We have the following diagram:

$$H_*(X) \xleftarrow{p_*} H_*(Z) \xrightarrow{q_*} H_*(Y). \tag{2.2}$$

Theorem 2.7. [2,4,1] Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$. The relations introduced in Definitions 2.4 and 2.5 satisfy the following conditions:

2.7.1 for each $x \in X$

$$(((p_1, q_1) \sim_k (p_2, q_2)) \;\text{ or }\; ((p_1, q_1) \sim_g (p_2, q_2))) \Rightarrow (q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x))),$$

2.7.2 $(((p_1, q_1) \sim_k (p_2, q_2))$ or $((p_1, q_1) \sim_g (p_2, q_2))) \Rightarrow (q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1})$,

2.7.3 Let $(p_3, q_3), (p_4, q_4) \in D(Y, T)$. Then

$$((p_1, q_1) \sim_k (p_2, q_2) \;\text{ i }\; (p_3, q_3) \sim_k (p_4, q_4)) \Rightarrow (((p_3, q_3) \circ (p_1, q_1)) \sim_k ((p_4, q_4) \circ (p_2, q_2))),$$

$$((p_1, q_1) \sim_g (p_2, q_2) \;\text{ i }\; (p_3, q_3) \sim_g (p_4, q_4)) \Rightarrow (((p_3, q_3) \circ (p_1, q_1)) \sim_g ((p_4, q_4) \circ (p_2, q_2))).$$

The set $M_k(X, Y) = D(X, Y)_{/\sim_k}$ will be called a k-morphism while the set $M_g(X, Y) = D(X, Y)_{/\sim_g}$ will be called a g-morphism. From Theorem 2.7 we get the following definition:

Definition 2.8. For any $\varphi_k \in M_k(X, Y)$ $(\varphi_g \in M_g(X, Y))$ the set $\varphi(x) = q(p^{-1}(x))$ where $\varphi_k = [(p, q)]_k$ $(\varphi_g = [(p, q)]_g)$ is called an image of the point x in the k-morphism φ_k $(g -$ morphism $\varphi_g)$ where $(p, q) \in D(X, Y)$.

A map $\varphi : X \multimap Y$ is compact, if $\overline{\varphi(X)} \subset Y$ is a compact set. Let $(p, q) \in D(X, X)$, where $p, q : Z \to X$. We say that p and q have a coincidence point if there exists a point $z \in Z$ such that $p(z) = q(z)$. Let $z \in Z$ and let $x = p(z)$. We observe that

$$(p(z) = q(z)) \Leftrightarrow (x \in q(p^{-1}(x))). \tag{2.3}$$

Theorem 2.9. [4] Let X be a metrizable space. Consider a diagram:

$$X \xleftarrow{p} Z \xrightarrow{q} X,$$

in which $X \in ANR$, p is Vietoris and q is compact. Then $q_* \circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* \circ p_*^{-1}) \neq 0$ implies that p and q have a coincidence point.

Definition 2.10. Let X and Y be metrizable spaces. We say that a continuous map $f : X \to Y$ is universal if for each continuous map $g : X \to Y$ f and g have a coincidence point.

Let $\mathbb{R}$ be a real number set and let $[0,1] \subset \mathbb{R}$ be an interval. Let

$$[0,1]^n = [0,1] \times [0,1] \times ... \times [0,1] \ (n - th \ [0,1]).$$

Theorem 2.11. (see [8,9]) Let X be a connected metrizable space. If there exists a universal map $f : X \to [0,1]^n$ then $dimX \geq n$.

3 Abstract Morphisms

In this paragraph we assume that all spaces are Hausdorff topological spaces. With the help of Theorem 2.7, we will introduce the notion of an abstract morphism.

Definition 3.1. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$. The equivalency relation in the set $D(X, Y)$ is called a constructor of morphisms (it is denoted as $\sim_a$), if the following conditions are satisfied:

3.1.1 $((p_1, q_1) \sim_a (p_2, q_2)) \Rightarrow$ (for each $x \in X$ $q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x)))$,

3.1.2 $((p_1, q_1) \sim_a (p_2, q_2)) \Rightarrow (q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1})$,

3.1.3 Let $(p_3, q_3), (p_4, q_4) \in D(Y, T)$. Then

$$((p_1, q_1) \sim_a (p_2, q_2) \text{ and } (p_3, q_3) \sim_a (p_4, q_4)) \Rightarrow (((p_3, q_3) \circ (p_1, q_1)) \sim_a ((p_4, q_4) \circ (p_2, q_2))).$$

The condition (3.1.1) will be called an axiom of topological equality, the condition (3.1.2) - an axiom of homological equality, and the condition (3.1.3) - an axiom of composition.

The set $M_a(X, Y) = D(X, Y)_{/\sim_a}$ will be called a set an abstract morphisms (a-morphisms). Definition 3.1 (condition 3.1.1) leads to the following:

Definition 3.2. Let $(p, q) \in D(X, Y)$. For any $\varphi_a \in M_a(X, Y)$ the set $\varphi(x) = q(p^{-1}(x))$ where $\varphi_a = [(p, q)]_a$ is called an image of the point x in the a-morphism φ_a. We denote by

$$\varphi : X \to_a Y \tag{3.1}$$

a multi-valued map determined by an a-morphism $\varphi_a = [(p, q)]_a \in M_a(X, Y)$ and will called an abstract multi-valued map.

The mapping $\varphi : X \multimap Y$ is called strongly admissible (see [4]) if there exists a diagram $(p, q) \in D(X, Y)$ such that for every $x \in X$

$$q(p^{-1}(x)) = \varphi(x). \tag{3.2}$$

Such a mapping can be represented by many a-morphisms. Let $\mathbb{S}^n$ denote a sphere in Euclidean space $\mathbb{R}^{n+1}$.

Example 3.3. Let $\varphi : \mathbb{S}^n \multimap \mathbb{S}^n$ be a strongly admissible mapping described as in the example (40.7) ([4], p. 202). Then there exist $(p_1, q_1), (p_2, q_2) \in D(\mathbb{S}^n, \mathbb{S}^n)$ such that for every $x \in \mathbb{S}^n$

$$q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x)) = \varphi(x), \text{but } q_{1*} \circ p_{1*}^{-1} \neq q_{2*} \circ p_{2*}^{-1}.$$

Hence, and from Definition 3.1 (3.1.2) it results that

$$(p_1, q_1) \nsim_a (p_2, q_2).$$

For single-valued mappings there is the following fact:

Proposition 3.4. Let $f : X \to Y$ be a continuous map and let $f_a \in M_a(X, Y)$ be an a-morphism such that for each $(p, q) \in f_a$ $q(p^{-1}(x)) = f(x)$ for each $x \in X$. Then $q = f \circ p$ for each $(p, q) \in f_a$.

Proof. Let $(p,q) \in f_a$. Then from the assumption we have for each $x \in X$ $q(p^{-1}(x)) = f(x)$, where $p : Z \to X$ is a Vietoris map and $q : Z \to Y$ is a continuous map. Let $z \in Z$. Then there exists a point $x_1 \in X$ such that $z \in p^{-1}(x_1)$. Hence we get

$$q(z) = f(x_1) = f(p(z)),$$

and the proof is complete. □

Let $\mathbf{TOP}$ denote categories in which Hausdorff topological spaces are objects and continuous mappings are category mappings. Let $\mathbf{TOP_a}$ denote categories in which Hausdorff topological spaces are objects and abstract multi-valued maps (see (3.1)) are category mappings. According to Definition 3.1 (3.1.3) the category of $\mathbf{TOP_a}$ is well defined and $\mathbf{TOP} \subset \mathbf{TOP_a}$. Let $\mathbf{VECT_G}$ denote categories in which linear graded vector spaces are objects and linear mappings of degree zero are category mappings.

Theorem 3.5. The mapping $\widetilde{\mathbf{H}_*} : \mathbf{TOP_a} \to \mathbf{VECT_G}$ given by the formula

$$\widetilde{\mathbf{H}_*}(\varphi) = q_* \circ p_*^{-1},$$

where φ is an abstract multi-valued map determined by $\varphi_a = [(p,q)]_a$ is a functor and the extension of the functor of the Čech homology $\mathbf{H}_* : \mathbf{TOP} \to \mathbf{VECT_G}$.

Proof. From the axiom of homological equality it results that the mapping $\widetilde{\mathbf{H}_*}$ is well defined. From Proposition 3.4 it results that if $\varphi : X \to_a X$ is an identity then $(p,p) \in \varphi_a$ where $p : Z \to X$ is some Vietoris mapping. Hence

$$\widetilde{\mathbf{H}_*}(\varphi) = p_* \circ p_*^{-1} = Id_*.$$

Let $\varphi : X \to_a Y$ and $\psi : Y \to_a T$ and let $(p_1, q_1) \in \varphi_a$ and $(p_2, q_2) \in \psi_a$. We have the following commutative diagrams (see Definition 2.3):

$$\begin{array}{ccccc} Z_1 & \xrightarrow{q_1} & Y & \xleftarrow{p_2} & Z_2 \\ \uparrow{\scriptstyle Id} & & & & \uparrow{\scriptstyle Id} \\ Z_1 & \xleftarrow{f_1} & Z_1 \triangle_{q_1 p_2} Z_2 & \xrightarrow{f_2} & Z_2 \end{array} \qquad \begin{array}{ccccc} H_*(Z_1) & \xrightarrow{q_{1*}} & H_*(Y) & \xleftarrow{p_{2*}} & H_*(Z_2) \\ \uparrow{\scriptstyle Id_*} & & & & \uparrow{\scriptstyle Id_*} \\ H_*(Z_1) & \xleftarrow{f_{1*}} & H_*(Z_1 \triangle_{q_1 p_2} Z_2) & \xrightarrow{f_{2*}} & H_*(Z_2), \end{array}$$

that is

$$p_2 \circ f_2 = q_1 \circ f_1 \quad \text{and} \quad p_{2*} \circ f_{2*} = q_{1*} \circ f_{1*}.$$

Hence

$$p_{2*}^{-1} \circ q_{1*} = f_{2*} \circ f_{1*}^{-1}.$$

We have:

$$\widetilde{\mathbf{H}_*}(\psi \circ \varphi) = (q_2 \circ f_2)_* \circ (p_1 \circ f_1)_*^{-1} = (q_2)_* \circ ((f_2)_* \circ (f_1)_*^{-1}) \circ (p_1)_*^{-1} =$$

$$= (q_2)_* \circ ((p_2)_*^{-1} \circ (q_1)_*) \circ (p_1)_*^{-1} = ((q_2)_* \circ (p_2)_*^{-1}) \circ ((q_1)_* \circ (p_1)_*^{-1}) = \widetilde{\mathbf{H}_*}(\psi) \circ \widetilde{\mathbf{H}_*}(\varphi).$$

It shall be noticed that if $f : X \to Y$ is a continuous function ($f \in \mathbf{TOP}$), then from Proposition 3.4 it results that if $(p,q) \in f_a$ then $q = f \circ p$. Hence what follows is

$$\widetilde{\mathbf{H}_*}(f) = q_* \circ p_*^{-1} = (f \circ p)_* \circ p_*^{-1} = (f_* \circ p_*) \circ p_*^{-1} = f_* \circ (p_* \circ p_*^{-1}) = f_* = \mathbf{H}_*(f)$$

and the proof is complete. □

From the point of view of Čech homologies, abstract multi-valued maps behave similarly to single-valued mappings.

Let $X_0 \subset X,\ Y_0 \subset Y$ and let $(p,q) \in D(X,Y)$ such that $q(p^{-1}(X_0)) \subset Y_0$. We shall denote by

$$\overline{p} : p^{-1}(X_0) \to X_0\ \overline{p}(z) = p(z),\ \ \overline{q} : p^{-1}(X_0) \to Y_0\ \overline{q}(z) = q(z) \text{ for all } z \in p^{-1}(X_0). \tag{3.3}$$

We observe that from Definition 2.3:

$$(\overline{p},\overline{q}) = (p,q)\circ(Id,i), \tag{3.4}$$

where

$$X_0 \xleftarrow{\ Id\ } X_0 \xrightarrow{\ i\ } X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y$$

$i : X_0 \to X$ is an inclusion given by formula $i(x) = x$ for each $x \in X_0$. From the axiom of composition (see Definition 3.1) and (3.4) we get the following fact:

Proposition 3.6. Let $X_0 \subset X,\ Y_0 \subset Y$ and let $(p_1,q_1),(p_2,q_2) \in D(X,Y)$ such that $q_1(p_1^{-1}(X_0)) \subset Y_0$ and $q_2(p_2^{-1}(X_0)) \subset Y_0$. Then $(\overline{p}_1,\overline{q}_1),(\overline{p}_2,\overline{q}_2) \in D(X_0,Y_0)$ (see 3.3) and

$$(p_1,q_1) \sim_a (p_2,q_2) \Rightarrow (\overline{p}_1,\overline{q}_1) \sim_a (\overline{p}_2,\overline{q}_2).$$

We recall that the mapping $\varphi : X \multimap Y$ is acyclic if for every $x \in X$ the set $\varphi(x)$ is compact and acyclic.

Remark 3.7. Each mapping strongly admissible (see (3.2)) in particular acyclic (see [4]) is determined by a some a-morphism.

4 Fixed points of Abstract Morphisms

In this paragraph we assume that all spaces are metrizable. Let $\psi : X \to_a X$ be an abstract multi-valued map determined by an a-morphism $\psi_a \in M_a(X,X)$. It shall be denoted as

$$\widetilde{\mathbf{H}}_*(\psi) = \psi_* \ \text{ (see Theorem 3.5)}$$

and if the homomorphism $\psi_* : H_*(X) \to H_*(X)$ is a Leray endomorphism, then the generalized Lefschetz number of ψ will be denoted by the symbol

$$\Lambda(\psi) = \Lambda(\psi_*).$$

Let $X_0 \subset X$. A map $\varphi : (X,X_0) \multimap (X,X_0)$ is an abstract multi-valued map if and only if the map $\varphi_X : X \multimap X$ given by

$$\varphi_X(x) = \varphi(x) \ \text{ for each } \ x \in X$$

is an abstract multi-valued map and $\varphi_X(X_0) \subset X_0$. Then from Proposition 3.6 the map $\varphi_{X_0} : X_0 \multimap X_0$ given by formula

$$\varphi_{X_0}(x) = \varphi(x) \ \text{ for each } \ x \in X_0$$

is an abstract multi-valued map. Let $(p,q) \in (\varphi_X)_a,\ p,q : Z \to X$. We shall denote by $\widetilde{p} : (Z,p^{-1}(X_0)) \to (X,X_0)$ $\widetilde{p}(z) = p(z)$, $\widetilde{q} : (Z,p^{-1}(X_0)) \to (X,X_0)$ $\widetilde{q}(z) = q(z)$ for all $z \in Z$. We observe that $(\widetilde{p},\widetilde{q}) \in \varphi_a$ and $(\overline{p},\overline{q}) \in (\varphi_{X_0})_a$ (see (3.3)). We have the following diagram:

$$H_*(X,X_0) \xleftarrow{\ \widetilde{p}_*\ } H_*(Z,p^{-1}(X_0)) \xrightarrow{\ \widetilde{q}_*\ } H_*(X,X_0), \tag{4.1}$$

where $\widetilde{p}_*$ is an isomorphism (see [4]). Assume that

$$\widetilde{q}_* \circ \widetilde{p}_*^{-1} : H_*(X,X_0) \to H_*(X,X_0) \tag{4.2}$$

is a Leray endomorphism. For such a φ, we define the Lefschetz number $\Lambda(\varphi)$ of φ by putting

$$\Lambda(\varphi) = \Lambda(\widetilde{q}_* \circ \widetilde{p}_*^{-1}). \tag{4.3}$$

The Lefschetz number of φ (see (4.3)) mapping is well defined. It is the result of Proposition 3.6 and the following well-known mathematical fact:

Proposition 4.1. (see [4]) Let $X_0 \subset X$ be a nonempty set and let $(p,q) \in D(X,X)$ such that

$q(p^{-1}(X_0)) \subset X_0$. If any two of endomorphisms $\widetilde{q}_* \circ \widetilde{p}_*^{-1} : H_*(X, X_0) \to H_*(X, X_0)$ (see (4.2)), $q_* \circ p_*^{-1} : H_*(X) \to H_*(X)$, $\overline{q}_* \circ \overline{p}_*^{-1} : H_*(X_0) \to H_*(X_0)$ (see (3.3)) are Leray endomorphisms, then so is the third and

$$\Lambda(\widetilde{q}_* \circ \widetilde{p}_*^{-1}) = \Lambda(q_* \circ p_*^{-1}) - \Lambda(\overline{q}_* \circ \overline{p}_*^{-1}).$$

From the axiom of homological equality, Proposition 4.1 and (4.3) we get the following fact:

Proposition 4.2. Let $\varphi : (X, X_0) \to_a (X, X_0)$ be an abstract multi-valued map. If any two of endomorphisms $\varphi_* : H_*(X, X_0) \to H_*(X, X_0)$, $(\varphi_X)_* : H_*(X) \to H_*(X)$, $(\varphi_{X_0})_* : H_*(X_0) \to H_*(X_0)$ are Leray endomorphisms, then so is the third and

$$\Lambda(\varphi) = \Lambda(\varphi_X) - \Lambda(\varphi_{X_0}).$$

Let $A \subset X$ be a nonempty set and let $\varphi : X \multimap Y$ be a map. We have a map $\varphi_A : A \multimap Y$ given by formula

$$\varphi_A(x) = \varphi(x) \text{ for each } x \in A. \tag{4.4}$$

Let $\varphi : X \multimap X$. The following mapping:

$$\varphi^n = \varphi \circ \varphi \circ ... \circ \varphi \ (n\text{-th } \varphi) \tag{4.5}$$

will be denoted by the symbol $\varphi^n : X \multimap X$ where n is a natural number. We recall the following definition:

Definition 4.3. We say that an u.s.c. map $\varphi : X \multimap X$ is a compact absorbing contraction (we write $\varphi \in CAC(X)$) if there exists an open set $U \subset X$ such that the following conditions are satisfied:
$4.3.1$ $\varphi_U : U \multimap U$ is a compact map (see (4.4)) ($\overline{\varphi_U(U)} \subset U$),
$4.3.2$ for each $x \in X$ there exists a natural number $n(x)$ such that $\varphi^{n(x)}(x) \subset U$ (see (4.5)).
From Proposition 42.2 (see [4], p. 209) we get the following result:

Proposition 4.4. If an abstract multi-valued map $\varphi_X \in CAC(X)$ then for every diagram $(p, q) \in (\varphi_X)_a$ the homomorphism

$$\varphi_* = \widetilde{q}_* \circ \widetilde{p}_*^{-1} : H_*(X, U) \to H_*(X, U) \text{ (see (4.2))}$$

is weakly nilpotent.

Theorem 4.5. Let $\varphi_X : X \to_a X$ be an abstract multi-valued map and let $X \in ANR$. Assume that $\varphi_X \in CAC(X)$ (see Definition 4.3). Then φ_X is a Leray endomorphism and if $\Lambda(\varphi_X) \neq 0$ then φ_X has a fixed point (there exists $x \in X$ such that $x \in \varphi_X(x)$).

Proof. We observe that $\varphi : (X, U) \multimap (X, U)$ given by formula

$$\varphi(x) = \varphi_X(x) \text{ for each } x \in X$$

is an abstract multi-valued map. From the axiom of homological equality, (4.3), Proposition 4.4 and Proposition 4.2, φ_* is weakly nilpotent and

$$\Lambda(\varphi) = 0.$$

The abstract multi-valued map φ_U is compact (for each $(p', q') \in (\varphi_U)_a$ q' is compact) and $U \in ANR$, so is a Leray endomorphism. Hence and from Proposition 4.2, $(\varphi_X)_*$ is a Leray endomorphism and

$$\Lambda(\varphi_X) = \Lambda(\varphi_U). \tag{4.6}$$

Assume that $\Lambda(\varphi_X) = \Lambda(q_* \circ p_*^{-1}) \neq 0$ for some diagram $(p, q) \in (\varphi_X)_a$, where $p : Z \to X$ is a Vietoris map and $q : Z \to X$ is continuous. Then from (4.6)

$$\Lambda(\varphi_U) = \Lambda(\overline{q}_* \circ \overline{p}_*^{-1}) \neq 0 \text{ (see (3.3), Proposition 3.6)}$$

and from Theorem 2.9 there exists a coincidence point $z \in p^{-1}(U) \subset Z$ such that

$$\overline{p}(z) = \overline{q}(z).$$

Hence $p(z) = q(z)$. From (2.3) for $p(z) = x \in U \subset X$ we have

$$x \in q(p^{-1}(x)) = \varphi_X(x) \text{ (the axiom of topological equality))}$$

and the proof is complete.

□

5 Other Examples of Abstract Morphisms

In this paragraph we assume that all spaces are metrizable. First, the following two facts should be proven:

Proposition 5.1. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$ and $(p_3, q_3), (p_4, q_4) \in D(Y, T)$. Assume that

$$(p, q) = (p_3, q_3) \circ (p_1, q_1) \text{ and } (r, s) = (p_4, q_4) \circ (p_2, q_2).$$

If the following diagrams:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Id} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y, \end{array} \qquad \begin{array}{ccccc} Y & \xleftarrow{p_3} & Z_3 & \xrightarrow{q_3} & T \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle Id} \\ Y & \xleftarrow{p_4} & Z_4 & \xrightarrow{q_4} & T \end{array}$$

are commutative, that is

$$p_2 \circ f = p_1, \ \ q_2 \circ f = q_1 \text{ and } p_4 \circ g = p_3, \ \ q_4 \circ g = q_3,$$

where f and g are single-valued maps (not necessarily continuous) then there exists a single-valued map $h : Z_1 \triangle_{q_1 p_3} Z_3 \to Z_2 \triangle_{q_2 p_4} Z_4$ (see Definition 2.3) such that the diagram:

$$\begin{array}{ccccc} X & \xleftarrow{p} & Z_1 \triangle_{q_1 p_3} Z_3 & \xrightarrow{q} & T \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Id} \\ X & \xleftarrow{r} & Z_2 \triangle_{q_2 p_4} Z_4 & \xrightarrow{s} & T \end{array}$$

is commutative.

Proof. We define a map

$$h : Z_1 \triangle_{q_1 p_3} Z_3 \to Z_2 \triangle_{q_2 p_4} Z_4 \tag{5.1}$$

given by formula

$$h(z_1, z_3) = (f(z_1), g(z_3)) \text{ for each } (z_1, z_3) \in Z_1 \triangle_{q_1 p_3} Z_3.$$

The map h is well defined. Let $(z_1, z_3) \in Z_1 \triangle_{q_1 p_3} Z_3$. We have

$$q_2(f(z_1)) = q_1(z_1) = p_3(z_3) = p_4(g(z_3)).$$

Hence $(f(z_1), g(z_3)) \in Z_2 \triangle_{q_2 p_4} Z_4$. We show that the diagram is commutative. Let $(z_1, z_3) \in Z_1 \triangle_{q_1 p_3} Z_3$ then (see Definition 2.3)

$$r(h(z_1, z_3)) = r(f(z_1), g(z_3)) = p_2(f(z_1)) = p_1(z_1) = p(z_1, z_3),$$

$$s(h(z_1, z_3)) = s(f(z_1), g(z_3)) = q_4(g(z_3)) = q_3(z_3) = q(z_1, z_3)$$

and the proof is complete. □

Proposition 5.2. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$. Assume that the following diagram:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Id} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y \end{array}$$

is commutative, that is

$$p_2 \circ f = p_1, \;\; q_2 \circ f = q_1,$$

where $f : Z_1 \to Z_2$ is a continuous map. Then the following conditions are satisfied:
$5.2.1$ for each $x \in X$ $q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x))$,
$5.2.2$ $q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1}$.

Proof. $5.2.1$ Let $x \in X$. We have

$$q_1(p_1^{-1}(x)) = (q_2 \circ f)((p_2 \circ f)^{-1}(x)) = q_2(f(f^{-1}(p_2^{-1}(x)))) = q_2(p_2^{-1}(x)).$$

$5.2.2$ We observe that from the assumption the homomorphism f_* is an isomorphism and

$$q_{1*} \circ p_{1*}^{-1} = (q_2 \circ f)_* \circ (p_2 \circ f)_*^{-1} = (q_{2*} \circ f_*) \circ (p_{2*} \circ f_*)^{-1} =$$
$$= (q_{2*} \circ f_*) \circ (f_*^{-1} \circ p_{2*}^{-1}) = q_{2*} \circ (f_* \circ f_*^{-1}) \circ p_{2*}^{-1} = q_{2*} \circ p_{2*}^{-1}$$

and the proof is complete. □

Of course many other abstract morphisms can be constructed, at least with regard to the kind of mappings used in their definition. It shall be reminded that the mapping $f : X \to Y$ satisfies the Lipschitz condition that if there exists a real number $L \geq 0$ such that for every $x, y \in X$

$$d_Y(f(x), f(y)) \leq L d_X(x, y),$$

where d_X, d_Y are metrics in spaces X, Y respectively.
Example 5.3. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$.

$$(p_1, q_1) \sim_L (p_2, q_2)$$

if and only if there exist spaces Z_1, Z_2 and the mappings satisfying the Lipschitz condition $g : Z_1 \to Z_2,\; h : Z_2 \to Z_1$ such that the following diagrams:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle Id} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y, \end{array} \qquad \begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle Id} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y, \end{array}$$

are commutative, that is

$$p_2 \circ g = p_1, \;\; q_2 \circ g = q_1 \;\text{ and }\; p_1 \circ h = p_2, \;\; q_1 \circ h = q_2.$$

It shall be noticed that ($\sim_L$) is an equivalence relation in the set $D(X, Y)$ because the identity map satisfies the Lipschitz condition (reflexivity) and the composition of two mappings satisfying the Lipschitz condition also satisfies the Lipschitz condition (transitivity). The symmetry is a direct result of the definition of the relation. It is easy to prove that ($\sim_L$) is the constructor of L-morphisms (see Proposition 5.1 and Proposition 5.2)

$$\varphi_L = [(p, q)]_L \in M_L(X, Y),$$

where $(p, q) \in D(X, Y)$.

Example 5.4. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$.

$$(p_1, q_1) \sim_V (p_2, q_2)$$

if and only if there exist spaces Z_1, Z_2 and the Vietoris mappings $p : Z_1 \to Z_2,\ p' : Z_2 \to Z_1$ such that the following diagrams:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle Id} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y, \end{array} \qquad \begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p'} & & \uparrow{\scriptstyle Id} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y, \end{array}$$

are commutative, that is

$$p_2 \circ p = p_1,\ \ q_2 \circ p = q_1 \ \text{ and } \ p_1 \circ p' = p_2,\ \ q_1 \circ p' = q_2.$$

The justification that $(\sim_V)$ is an equivalence relation in the set $D(X, Y)$ is similar to the justification from Example 5.3. The relation is obviously the constructor of V-morphisms (see Proposition 5.1 and Proposition 5.2)

$$\varphi_V = [(p, q)]_V \in M_V(X, Y),$$

where $(p, q) \in D(X, Y)$.
Metric spaces X and Y are isometric if there exists a surjection $f : X \to Y$ (isometry) such that

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2),\ \text{ for each }\ x_1, x_2 \in X,$$

where d_X, d_Y are metrics of spaces X and Y respectively.
Example 5.5. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$.

$$(p_1, q_1) \sim_I (p_2, q_2)$$

if and only if there exist isometric spaces Z_1, Z_2 and an isometry $g : Z_1 \to Z_2$ such that the following diagram:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle Id} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y, \end{array}$$

is commutative, that is

$$p_2 \circ g = p_1,\ \ q_2 \circ g = q_1.$$

It is clear that the isometry is a homeomorphism and the mapping inverse to an isometry is also an isometry. Hence we get:

$$p_1 \circ g^{-1} = p_2,\ \ q_1 \circ g^{-1} = q_2, \tag{5.2}$$

where $g^{-1} : Z_2 \to Z_1$ is an isometry inverse to g. From (5.2) results the symmetry of the relation. Reflexivity and transitivity can be justified in a similar way to how it was done in the previous examples. It is clear that the relation is the constructor of I-morphisms (see Proposition 5.1 and Proposition 5.2)

$$\varphi_I = [(p, q)]_I \in M_I(X, Y),$$

where $(p, q) \in D(X, Y)$.
It shall be noticed that for $(p, q) \in D(X, Y)$, we get the following:

$$[(p, q)]_I \subset [(p, q)]_k \subset [(p, q)]_V \ \text{ and } \ [(p, q)]_I \subset [(p, q)]_L \subset [(p, q)]_g.$$

Let $[0, 1] \subset \mathbb{R}$ be an interval in the set of real numbers $\mathbb{R}$. The following example will be given by the end of this article now (see [4,2]):

Example 5.6. Let $f : [0,1] \to [0,1]$ be a map given by formula $f(x) = 1$ for each $x \in [0,1]$. We have commutative diagrams:

$$\begin{array}{ccccc} [0,1] & \xleftarrow{p_f} & \Gamma_f & \xrightarrow{q_f} & [0,1] \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Id} \\ [0,1] & \xleftarrow{p_1} & [0,1]\times[0,1] & \xrightarrow{q_1} & [0,1], \end{array} \qquad \begin{array}{ccccc} [0,1] & \xleftarrow{p_f} & \Gamma_f & \xrightarrow{q_f} & [0,1] \\ \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle Id} \\ [0,1] & \xleftarrow{p_1} & [0,1]\times[0,1] & \xrightarrow{q_1} & [0,1], \end{array}$$

where

$$\Gamma_f = \{(x,y) \in [0,1] \times [0,1] : y = f(x)\} \approx [0,1] \times \{1\},$$

$p_f(x,1) = x$, $q_f(x,1) = 1$, $p_1(x,y) = x$, $q_1(x,y) = 1$, $h(x,1) = (x,1)$ and $g(x,y) = (x,1)$ for each $(x,y) \in [0,1] \times [0,1]$. We observe that

$$(p_f, q_f) \sim_L (p_1, q_1)$$

(the mappings g and h satisfy the Lipschitz condition with the constant $L = 1$)

$(p_f, q_f) \nsim_I (p_1, q_1)$ (spaces $[0,1]$ and $[0,1] \times [0,1]$ are not isometric),

$$(p_f, q_f) \nsim_V (p_1, q_1)$$

(there does not exist a Vietoris mapping $[0,1] \to [0,1] \times [0,1]$ $(see\ Theorems\ 2.9,\ 2.11)$).

Remark 5.7. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$. The poorest relation of equivalence (the classes of abstracts consist of a single element) that is the constructor of Id-morphisms (Id is an identity map) is a relation defined in the following way:

$$((p_1, q_1) \sim_{Id} (p_2, q_2)) \Leftrightarrow (p_2 \circ Id = p_1 \text{ and } q_2 \circ Id = q_1).$$

It can be proven that the relation of equivalence given by the formula:

$$((p_1, q_1) \sim_A (p_2, q_2)) \Leftrightarrow$$

$$\Leftrightarrow ((\text{for each } x \in X \;\; q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x))) \text{ and } (q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1}))$$

is the constructor of absolute morphisms (A-morphisms) and its classes of abstracts encompass all diagrams that satisfy the axioms of topological and homological equivalence.

6 Conclusions

Abstract morphisms can be created according to where and how they are to be applied. They will be used in many different fields such as: fixed point theory, differential inclusion, or theory of multi-domination (see [10]).

Competing Interests

The author declares that no competing interests exist.

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