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A Note on Gordan's Theorem

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Abstract

Inspired by Gale's proof of Farkas's lemma, this expository note aims to give a very simple and intuitive proof of Gordan's theorem and the equivalence of this theorem to Farkas's lemma and some formulations of the separating hyperplane theorems. The tool we have employed is limited to only very simple linear algebra over the field of real numbers.

Keywords: Gordan's theorem; Farkas's lemma; separating hyperplane theorem; Stiemke's theorem.

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1 Introduction

Gordan's theorem was published in 1873 [1], many years before another equivalent version, namely Farkas's lemma [2], which was published around the year 1900. These theorems have important implications to other areas such as linear programming and economics. It is not our goal to introduce the full history here, but the interested reader may refer to [3] for an account of history regarding Farkas's lemma, and to [4] for other related theorems, such as Stiemke's theorem [5] (a.k.a. the Fundamental Theorem of Asset Pricing [6]), Motzkin's theorem [7], Gale's theorem [8], and von Neumann's mini-max theorem [9]. In [3], the author already mentioned the equivalence of many of such theorems, and he gave an analogy that these theorems are like the cities on a plateau: it is comparatively easier to travel from one city to the other, but to reach one of these cities, one needs to climb up to the plateau. In [10], the author referred to introducing Farkas's lemma as a

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"pedagogical annoyance" because some parts of it are easy to verify while the main result cannot be proved in an elementary way. Earlier simple algebraic proof was given by Gale [8], but he mentioned that the proof is rather formal and doesn't make it clear why the theorem works. Motivated by the above comments and inspired by Gale's algebraic proof, we offer here a simple and intuitive (i.e. with more geometric flavor) proof of Gordan's theorem. Furthermore while economists [11] would give proofs for versions of separating hyperplane theorems based on intuition from economics and pricing, mathematicians would most likely dismiss such economics theorems as consequence of the separating hyperplane theorems. Hence we take the opportunity here to state some versions of the separating hyperplane theorems, and we prove that these separating hyperplane theorems are in fact equivalent to Gordan's theorem and Farkas's lemma. Since these theorems are widely applied in many different areas, we hope that our results deliver some level of simplicity and clarity, and being self-contained, they might serve some pedagogical purpose.

2 Preliminaries

We intend to formulate all theorems in a geometric language. So let's start with the following definition.

Definition 2.1. A linear polyhedral cone in $\mathbb{R}^n$ is a set V generated by nonzero vectors $v_1, \cdots, v_m \in \mathbb{R}^n$ with nonnegative coefficients, i.e.

$$V = \{\alpha_1 v_1 + \cdots + \alpha_m v_m | \alpha_i \geq 0, i = 1, \cdots, m\},$$

denoted by $V =< v_1, \cdots, v_m >_+$. We say that V is *pointed* if

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = 0, \alpha_i \geq 0, i = 1, \cdots, m \Rightarrow \alpha_1 = \cdots = \alpha_m = 0.$$

Remark 2.1. It is easy to show that V is pointed if and only if V does not contain a line.

We now distinguish ourselves with the following three versions of separating hyperplane theorems.

Theorem A. (Supporting Hyperplane Theorem or Gordan's Theorem) Let $V =< v_1, \cdots, v_m >_+$ be a pointed linear polyhedral cone. Then there exists y such that

$$y \cdot v_i < 0, 1 \leq i \leq m.$$

In this case, we say that the hyperplane H defined by $H = \{v | v \cdot y = 0\}$ is a *supporting hyperplane* for V at $\{0\}$.

Theorem B. (Separation I) Let $V =< v_1, \cdots, v_m >_+$ be a pointed linear polyhedral cone and S be a vector subspace (of $\mathbb{R}^n$) such that $V \cap S = \{0\}$. Then there exists a hyperplane H containing S such that H forms a supporting hyperplane for V at $\{0\}$.

Theorem C. (Separation II) Let V_1 and V_2 be two pointed linear polyhedral cones such that $V_1 \cap V_2 = \{0\}$. Then there exists y such that $y \cdot v < 0$ for all nonzero $v \in V_1$ and $y \cdot v > 0$ for all nonzero $v \in V_2$.

Remark 2.2. We mention here Gordan's original formulation of Theorem A (Gordan's Theorem). Let S be a system of linear equations with real coefficients in the unknowns $x_1, \cdots, x_m$. We say that $(x_1, \cdots, x_m)$ is a *semi-positive* solution of S if $(x_1, \cdots, x_m)$ satisfies the system and $x_i \geq 0, i = 1, \cdots, m$ but not all zero. Clearly a linear equation $F := A_1 x_1 + \cdots + A_m x_m = 0$ has no semi-positive solutions if all the coefficients A_i's are positive. Conversely, Gordan [1] showed that if S has no semi-positive solutions, then S leads to a linear equation of the above form $F = 0$. For completeness, we indicate below that this latter formulation follows directly from Theorem A.

Proof. Let S be described by the system $AX = 0$, where A is an $n \times m$ matrix and $X = (x_1, \cdots, x_m)^T$. If S has no semi-positive solutions, then the linear polyhedral cone formed by the column vectors $v_1, \cdots, v_m$ of A is pointed (in $\mathbb{R}^n$). Now Theorem A implies that there exists $y \in \mathbb{R}^n$ such that $y \cdot v_i < 0, i = 1, \cdots, m$. This shows that $-y^T A = (A_1, \cdots, A_m)$, where A_i's are all positive. Then $F := -y^T AX = 0$ yields the required linear equation. $\square$

Remark 2.3. Theorem B clearly implies Stiemke's theorem, and vice versa (see e.g. [6], [12], [13] for details).

Remark 2.4. Theorem A and Theorem C are equivalent.

Proof. It is trivial to see that Theorem C implies Theorem A. Conversely, let $V_1 =< u_1, \cdots, u_m >_+$ and $V_2 =< v_1, \cdots, v_r >_+$ be pointed linear polyhedral cones, and $V_1 \cap V_2 = \{0\}$. Consider $V =< u_1, \cdots, u_m, -v_1, \cdots, -v_r >_+$. Then it is easy to see that V is pointed, since

$$\alpha u_1 + \cdots + \alpha_m u_m + \beta_1(-v_1) + \cdots + \beta_r(-v_r) = 0, \alpha_1 \geq 0, \cdots, \alpha_m \geq 0, \beta_1 \geq 0, \cdots, \beta_r \geq 0$$

$$\Rightarrow \alpha_1 u_1 + \cdots + \alpha_m u_m = \beta_1 v_1 + \cdots + \beta_r v_r \in V_1 \cap V_2 = \{0\}$$

$$\Rightarrow \alpha_1 = \cdots = \alpha_m = 0 \text{ and } \beta_1 = \cdots = \beta_r = 0.$$

By Theorem A, there exists y such that $y \cdot u_i < 0, 1 \leq i \leq m$ and $y \cdot (-v_i) < 0, 1 \leq i \leq r$. This is the same as saying $y \cdot v < 0$ for all nonzero $v \in V_1$, and $y \cdot v > 0$ for all nonzero $v \in V_2$. $\square$

We now state

Farkas's Lemma. Let V be a linear polyhedral cone in $\mathbb{R}^n$ and $b \notin V$ be a vector in $\mathbb{R}^n$. Then there exists a vector y such that $y \cdot v \leq 0$ for all $v \in V$ and $y \cdot b > 0$.

We will prove Theorem A (Gordan's theorem) in section 3, and the equivalence of Theorem A, Theorem B (Separation I) and Farkas's lemma in section 4.

3 Geometric Proof of Gordan's Theorem

Proof. We prove by induction on the number m of the generators of the pointed polyhedral cone, the case of $m = 1$ being trivial. Assuming the case $m - 1$ ($m \geq 2$), we move on to show the case of m. Suppose we are given the generators $\{v_1, v_2, \cdots, v_m\}$ of V.

By induction hypothesis applied to the cone generated by $\{v_1, \cdots, v_{m-1}\}$, there is a vector y_1 satisfying $y_1 \cdot v_i < 0, \forall i < m$. If $y_1 \cdot v_m < 0$, we would be finished. Hence we just need to deal with the following two subcases:

Subcase 1: $y_1 \cdot v_m = 0$. In this case, we perturb y_1 by replacing it with $y' = y_1 - \epsilon v_m$. If $\epsilon > 0$ is small enough, we will still have $y' \cdot v_i < 0, \forall i < m$, however $y' \cdot v_m < 0$. This concludes Subcase 1.

Subcase 2: $y_1 \cdot v_m > 0$. In this case, the set of vectors $\{v_1, \cdots, v_{m-1}\}$ and the vector v_m are separated by the hyperplane H with normal vector y_1. For each of the v_i's with $i < m$, the polyhedral cone generated by v_i and v_m intersects H at a ray (note that v_i and v_m are necessarily linearly independent by the assumption on V). For each $i < m$ and on the corresponding ray, we can have a vector $v'_i \in H$ of the form $v'_i = v_i + a_i v_m$ for some $a_i > 0$. It is clear that $< v'_1, \cdots, v'_{m-1} >_+$ satisfies the assumption of Gordan's theorem, otherwise, we will have a relation of $v_1, \cdots, v_m$ with semi-positive coefficients: namely a relation $\sum_{i=1}^{m-1} c_i v'_i = 0$ with c_i's nonnegative and not all zero implies $\sum_{i=1}^{m-1} c_i(v_i + a_i v_m) = \sum_{i=1}^{m-1} c_i v_i + (\sum_{j=1}^{m-1} c_j a_j) v_m = 0$, contradicting the assumption of Gordan's theorem for $< v_1, \cdots, v_m >_+$. Hence by induction hypothesis applied to $\{v'_1, \cdots, v'_{m-1}\}$, there exists y' such that $y' \cdot v'_i < 0, \forall i < m$. Note that adding a multiple of y_1 to y' (i.e. replacing

y' by $y' + cy_1$) does not change the condition $y' \cdot v_i' < 0$ (since $y_1 \cdot v_i' = 0$), while we can do this in such a way that the resulting y' satisfies further the condition $y' \cdot v_m = 0$: to achieve this, solve $(y' + cy_1) \cdot v_m = 0$ for c, and replace y' by $y' + cy_1$.

But now, we have $y' \cdot v_i = y' \cdot (v_i' - a_i v_m) = y' \cdot v_i' < 0, \forall i < m$ and $y' \cdot v_m = 0$, hence we have reduced to Subcase 1. □

Remark 3.1. Gordan's theorem and Farkas's lemma stay true when the base field is replaced by linearly order (skew-)field (see [14] and [15] for more details).

4 Equivalence of Gordan's Theorem, Separation I and Farkas's Lemma

Farkas ⇒ Gordan

Proof. Let $V =< v_1, \cdots, v_m >_+$ be pointed. Consequently $-v_i \notin V$ for each i.

By Farkas's Lemma applied to V and $-v_i$, there exists y_i such that $y_i \cdot v_j \le 0$ for $1 \le j \le m$ and $y_i \cdot (-v_i) > 0$. This implies, for each i, $y_i \cdot v_j \le 0$ for all $i \ne j$ but $y_i \cdot v_i < 0$.

Letting $y := y_1 + \cdots + y_m$, it is clear that $y \cdot v_i < 0$ for $1 \le i \le m$. □

Gordan ⇒ Separation I.

Proof. Let $V =< v_1, \cdots, v_m >_+ \subseteq \mathbb{R}^n$ be pointed and $S \subseteq \mathbb{R}^n$ a subspace such that $V \cap S = \{0\}$. In particular, V satisfies the assumption of Gordan's theorem: $\sum_{i=1}^m \alpha_i v_i = 0, \alpha_i \ge 0 \Rightarrow \alpha_i = 0 \ \forall i$.

Let $\{u_1, \cdots, u_s\}$ be a basis of S (we may assume $\dim(S) = s > 0$ otherwise the result is trivial). Let L (by abuse of notation, we regard this also as the matrix representing the linear transformation) be any invertible linear transformation mapping u_j to $\begin{bmatrix} 0 \\ e_j \end{bmatrix}$ for each $j, 1 \le j \le s$, where e_j's form the standard basis of $\mathbb{R}^s$.

Let $Lv_i = \begin{bmatrix} v_i' \\ s_i \end{bmatrix}, 1 \le i \le m$, where $v_i' \in \mathbb{R}^{n-s}$. We now show that v_i''s satisfy the assumption of Gordan's theorem: if $\sum \alpha_i v_i' = 0, \alpha_i \ge 0 \ \forall i$, then $\sum \alpha_i (Lv_i)$ is of the form $\begin{bmatrix} 0 \\ u \end{bmatrix} \in LV \cap LS = \{0\}$. Thus

$$\sum \alpha_i (Lv_i) = 0 \Rightarrow L(\sum \alpha_i v_i) = 0$$
$$\Rightarrow \sum \alpha_i v_i = 0 \Rightarrow \alpha_i = 0, \ \forall i.$$

By Gordan's theorem applied to $< v_1', \cdots, v_m' >_+$, there exists $y' \in \mathbb{R}^{n-s}$ such that $y' \cdot v_i' < 0, \ \forall i$. Therefore

$$\begin{bmatrix} y' \\ 0 \end{bmatrix} \cdot \begin{bmatrix} v_i' \\ s_i \end{bmatrix} < 0, 1 \le i \le m, \text{ and } \begin{bmatrix} y' \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ e_j \end{bmatrix} = 0, \ 1 \le j \le s.$$

In terms of matrices (denoting also by V the matrix formed by v_i's and by S the matrix formed by u_i's), the above relations can be written as $[y'^T \ 0]LV < 0$ and $[y'^T \ 0]LS = 0$. Letting $y := L^T \begin{bmatrix} y' \\ 0 \end{bmatrix}$, we see that $y^T V < 0$ and $y^T S = 0$. Namely, the hyperplane H with normal vector y contains S and forms a supporting hyperplane for V at $\{0\}$. □

Separation I ⇒ Farkas

Lemma 4.1. *Let $V =< v_1, \cdots, v_m >_+$ be a polyhedral cone. Then up to reordering, we may assume that the first r vectors (not necessarily linearly independent) generate $W := V \cap (-V)$ over the real numbers and the remaining $m - r$ vectors generate a pointed polyhedral cone V' over nonnegative real numbers, and we have a decomposition $V = W + V'$ with $W \cap V' = \{0\}$, where $W + V' := \{w + v' | w \in W$ and $v' \in V'\}$.*

Proof. (of Lemma 4.1) It suffices to observe that if $\alpha_1 v_{i_1} + \cdots + \alpha_k v_{i_k} = 0$ such that $\alpha_1 > 0, \cdots, \alpha_k > 0$, then each of $v_{i_1}, \cdots, v_{i_k}$ belongs to $V \cap (-V) = W$. Conversely, if $v_i \in W$, then $-v_i \in V$, i.e. $-v_i = \sum_{i=1}^{m} c_j v_j$, where $c_j \geq 0$, not all zero, whence v_i satisfies a relation of the form $\alpha_1 v_{i_1} + \cdots + \alpha_k v_{i_k} = 0$ with positive coefficients.

Proof of "Separation I ⇒ Farkas". Let $V =< v_1, \cdots, v_m >_+$ and $b \notin V$, we need to find y such that $y \cdot v_i \leq 0$ and $y \cdot b > 0$.

Consider $V_1 :=< v_1, \cdots, v_m, -b >_+$ and use Lemma 4.1 to decompose $V_1 = W + V_1'$. Note that $-b \notin W$: if $-b \in W$, then as in the proof of Lemma 4.1, there is a relation $\sum_{i \in \Lambda} \alpha_i v_i + (-b) = 0, \alpha_i > 0 \Rightarrow b = \sum_{i \in \Lambda} \alpha_i v_i \in V$, a contradiction. By Separation I applied to V_1' and W, there exists y such that $y \cdot w = 0$ for all $w \in W$ and $y \cdot v < 0$ for all $v \in V_1'$. Since $-b \in V_1'$, we have $y \cdot b > 0$. Similarly by Separation I, $y \cdot v_i = 0$ or $y \cdot v_i < 0$ according as $v_i \in W$ or $v_i \in V_1'$, therefore $y \cdot v \leq 0$ for all $v \in V$. □

5 Conclusion

Most of the theorems in this note can be recast in the format of theorem of alternatives (see [4] for more details). In this short note, we have singled out a few theorems and proved their equivalence; in fact more is true: many other related theorems are equivalent to these theorems: these were observed in [12], and in [13] for an expanded version.

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Competing Interests

The author declares that no competing interests exist.

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